Modeling Friction and Contact in Chrono

Theoretical Background
Things Covered

• Friction and contact, understanding the problem at hand

• The penalty approach

• The complementarity approach
Mass $\times$ Acceleration = Force
Mass × Acceleration = Force

- Coulomb friction coefficient - \( \mu \)

\[
m \ddot{v} = W + F + F_f + N
\]

\[
||F_f|| \leq \mu ||N|| 
\]

Reflect on this: friction force can assume a bunch of values (as long as they’re smaller than \( \mu \times N \) though)
Additive Manufacturing (3D SLS Printing)

Courtesy of Professor Tim Osswald, Polymer Engineering Center, UW-Madison
Two main approaches: penalty & complementarity

Problem

- Computational many-body dynamics
  - Handling frictional contact

Modelling approach

- Penalty-based approach
- Complementarity approach

Numerical techniques

- Collision detection
- Optimization techniques
General Comments, Penalty Approach

• Approach commonly used in handling granular material
  • Called “Discrete Element Method”

• The “Penalty” approach works well for sphere-to-sphere and sphere-to-plane scenarios
  • Deformable body mechanics used to characterize what happens under these scenarios

• Methodology subsequently grafted to general dynamics problem of rigid bodies – arbitrary geometry
  • When they collide, a fictitious spring-damper element is placed between the two bodies
    • Sometimes spring & damping coefficient based on continuum theory mentioned above
    • Sometimes values are guessed (calibration) based on experimental data
The Penalty Method, Taxonomy

• Depending on the normal relative velocity between bodies that experience a collision and their material properties, if there is no relative angular velocity, the collision is
  • Elastic, if the contact induced deformation is reversible and independent of displacement rate
  • Viscoelastic, if the contact induced deformation is irreversible, but the deformation is dependent on the displacement rate
  • Plastic, if collision leaves an involved body permanently deformed but the deformation of body is independent of the displacement rate
  • Viscoplastic, if impact is irreversible and similar to the viscoelastic contact but deformation depends on the displacement rate

• According to the dependency of the normal force on the overlap and the displacement rate, the force schemes can be subdivided into
  • Continuous potential models (like Lennard-Jones, for instance)
  • Linear viscoelastic models (simple, used extensively, what we use here)
  • Non-linear viscoelastic models
  • Hysteretic models (see papers of L. Vu-Quoc, in “DEM Further Reading” slide)
The Penalty Method in Chrono, Nuts and Bolts

- Method relies on a record (history) of tangential displacement $\delta_t$ to model static friction (see figure at right)
The Penalty Method in Chrono, Nuts and Bolts

\[
F_n = f \left( \frac{\delta_n}{D_{\text{eff}}} \right) \left( k_n \delta_n n - \gamma_n m_{\text{eff}} v_n \right)
\]

\[
F_t = f \left( \frac{\delta_n}{D_{\text{eff}}} \right) \left( -k_t \delta_t - \gamma_t m_{\text{eff}} v_t \right)
\]

If \( |F_t| > \mu |F_n| \) then scale \( |\delta_t| \) so that \( |F_t| = \mu |F_n| \)

Visualize this \( \delta_t \) as creep.
Direct Shear Analysis via Granular Dynamics
[using LAMMPS/LIGGGHTS and Chrono]

- 1800 uniform spheres randomly packed
- Particle Diameter: \( D = 5 \) mm
- Shear Speed: \( 1 \) mm/s
- Inter-Particle Coulomb Friction Coefficient: \( \mu = 0.5 \) (Quartz on Quartz)
- Void Ratio (dense packing): \( e = 0.4 \)
Direct Shear Analysis via Granular Dynamics
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Wave propagation in ordered granular material
Penalty Method – the Pros

• Backed by large body of literature and numerous validation studies

• No increase in the size of the problem
  • This is unlike the “complementarity” approach, discussed next

• Can accommodate shock wave propagation
  • Can’t do w/ “complementarity” approach since it’s a pure “rigid body” solution

• Easy to implement
  • Entire numerical solution decoupled
    • Easy to scale up to large problems
    • Parallel-computing friendly – run in parallel on per contact basis
      • Memory communication intensive
Penalty Method – Cons

1. Numerical stability requires small integration time steps
   • Long simulation times

2. Choice of integration time step strongly influences results

3. Sensitive wrt information provided by the collision detection engine

4. There is some hand-waving when it comes to arbitrary shapes and the fact that the friction force is a multi-valued function
DEM, Further Reading


The “Complementarity” Approach
aka
Differential Variational Inequality (DVI) Method
Two Shapes, and the Distance [Gap Function]

- Notation: \( \partial A \) represents set of points making up the boundary of body A
- Shape body A: collection of points \( S \) with \( \mathbf{r}_A^S = \mathbf{r}_A + \mathbf{A}_A \mathbf{s}_A^S, \quad \mathbf{s}_A^S \in \partial A \)
- Shape body B: collection of points \( S \) with \( \mathbf{r}_B^S = \mathbf{r}_B + \mathbf{A}_B \mathbf{s}_B^S, \quad \mathbf{s}_B^S \in \partial B \)

- Signed distance function in a given configuration \( \mathbf{q}_A \) and \( \mathbf{q}_B \)

\[
\Phi(\mathbf{q}_A(t), \mathbf{q}_B(t)) \equiv \min_{\mathbf{s}_A^S \in \partial A, \mathbf{s}_B^S \in \partial B} ||\mathbf{r}_A^S - \mathbf{r}_B^S||_2
\]

- Contact when distance function is zero

\[
\Phi(\mathbf{q}_A(t^*), \mathbf{q}_B(t^*)) = 0
\]
Body A – Body B Contact Scenario
Defining the Normal and Tangential Forces

- When a contact occurs: point of contact and local reference frame identified. Latter defined as follows:
  - \( \mathbf{u}_i \) and \( \mathbf{w}_i \) are two mutually perpendicular unit vectors in the tangent plan at the contact point
  - Unit vector \( \mathbf{n}_i \) defines the normal direction in the local reference frame

- A normal force appears along the direction normal to the plane of contact
  - Magnitude of the force is \( \hat{\gamma}_{i,n} \). Specifically,
    \[ \mathbf{F}_{i,N} = \hat{\gamma}_{i,n} \mathbf{n}_i \]

- A friction force appears in the tangent plane
  - Has two components along the axes \( \mathbf{u}_i \) and \( \mathbf{w}_i \): \( \hat{\gamma}_{i,u} \) and \( \hat{\gamma}_{i,w} \), respectively. Specifically,
    \[ \mathbf{F}_{i,T} = \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i \]

- NOTE: The point of contact, \( \mathbf{n}_i \), \( \mathbf{u}_i \), and \( \mathbf{w}_i \) are obtained at the end of the collision detection task, which is executed at the beginning of each time step
DVI-Based Methods: The Contact Model

- A contact is modeled by one inequality constraints, which states that either the distance between two bodies is greater than zero $\Phi_i(q) > 0$, in which case the normal force is zero $\hat{\gamma}_{i,n} = 0$, or vice-versa; i.e., if the distance is zero, the contact force is nonzero.

  - Condition above captured in the following complementarity condition:

    $$\hat{\gamma}_{i,n} \geq 0, \quad \Phi_i(q) \geq 0, \quad \Phi_i(q)\hat{\gamma}_{i,n} = 0,$$

  - Another way to state the complementarity condition:

    $$0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(q) \geq 0$$
DVI-Based Methods: The Friction Model

- The friction model considered is Coulomb’s:

\[ \mu_i \hat{\gamma}_{i,n} \geq \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \]

\[ \mathbf{F}_{i,T}^T \cdot \mathbf{v}_{i,T} = -\|\mathbf{F}_{i,T}\| \|\mathbf{v}_{i,T}\| \]

\[ \|\mathbf{v}_{i,T}\| \left( \mu_i \hat{\gamma}_{i,n} - \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \right) = 0 \]

- First condition: friction force is within the friction cone

- Second condition: friction force and tangential velocity between two bodies at point of contact are collinear and of opposite direction

- The third condition captures the stick-slip condition. If the velocity is greater than zero, it means that the friction force saturated; i.e., \( \mu_i \hat{\gamma}_{i,n} = \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \); this is the sliding scenario. Conversely, if the bodies stick to each other, then the relative tangential velocity is zero, \( \mathbf{v}_{i,T} = \mathbf{0}_3 \), and the friction force is not saturated \( \mu_i \hat{\gamma}_{i,n} > \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \).
Coulomb’s Model Posed as the Solution of an Optimization Problem

- Assume that \( \hat{\gamma}_{i,n} \) and \( v_{i,T} \) are given and you pose the following optimization problem in variables \( x \) and \( y \):
  - Minimize the function \( v_{i,T}^T (xu_i + yw_i) \) subject to the constraint \( \sqrt{x^2 + y^2} \leq \mu_i \hat{\gamma}_{i,n} \)

- If you pose the first order Karush-Kuhn-Tucker optimality conditions for this optimization problem you end up precisely with the set of three conditions that define the Coulomb friction model

- It follows that there is an interplay between \( \hat{\gamma}_{i,n} \), \( \hat{\gamma}_{i,u} \), \( \hat{\gamma}_{i,w} \), and \( v_{i,T} \). Using math notation

\[
(\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \operatorname{argmin}_{\sqrt{x^2 + y^2} \leq \mu_i \hat{\gamma}_{i,n}} v_{i,T}^T (xu_i + yw_i).
\]
The DVI Problem: The EOM, in Fine-Granularity Form

- Time evolution of the dynamical system is the solution of the following DVI problem:

\[
B = 1, \ldots, nb : \quad m_B \ddot{\mathbf{r}}_B = \sum_{i \in B(B)} \left[ \Psi_{i_B}^{(i)} \right]^T \hat{\gamma}_{i,b} + f_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in A(B)} (\hat{\gamma}_{i,n} \mathbf{n}_i + \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i)
\]

\[
B = 1, \ldots, nb : \quad \ddot{\mathbf{J}}_B \dot{\mathbf{\omega}}_B = \sum_{i \in B(B)} \Pi_B^T(\Psi^{(i)})(\hat{\gamma}_{i,b}) + \tau_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in A(B)} \ddot{s}_{i,B} A_B^T(\hat{\gamma}_{i,n} \mathbf{n}_i + \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i)
\]

\[
B = 1, \ldots, nb : \quad \dot{\mathbf{p}}_B = \frac{1}{2} \mathbf{G}^T(\mathbf{p}_B) \dot{\mathbf{\omega}}_B
\]

\[
i \in B : \quad \Psi_i(\mathbf{q}, t) = 0
\]

\[
i \in A : \quad 0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \geq 0,
\]

\[
i \in A : \quad (\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \arg\min_{\sqrt{x^2+y^2}\leq \mu_i \hat{\gamma}_{i,n}} \mathbf{v}^T (x \mathbf{D}_{i,u} + y \mathbf{D}_{i,w})
\]
Frictional Contact: The Matrix-Vector Form

- Problem on previous slide reformulated using matrix-vector notation, assumes form

\[
\dot{\mathbf{q}} = \mathbf{L}(\mathbf{q}) \mathbf{v} \\
\mathbf{M} \dot{\mathbf{v}} = \mathbf{f}(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in \mathcal{B}} \hat{\gamma}_{i,b} \nabla \Psi_i + \sum_{i \in \mathcal{A}} (\hat{\gamma}_{i,n} \mathbf{D}_{i,n} + \hat{\gamma}_{i,u} \mathbf{D}_{i,u} + \hat{\gamma}_{i,w} \mathbf{D}_{i,w})
\]

\[
i \in \mathcal{B} : \quad \Psi_i(\mathbf{q}, t) = 0
\]

\[
i \in \mathcal{A} : \quad 0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \geq 0,
\]

\[
(\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \arginf_{\sqrt{x^2 + y^2} \leq \mu_i \hat{\gamma}_{i,n}} v^T(x \mathbf{D}_{i,u} + y \mathbf{D}_{i,w})
\]
The Discretization Process

- For straight index-3 DAE solution (like ADAMS), one uses the Newton-Euler form of the equations of motion in conjunction with the level zero constraints (the position constraint equations)

- The DVI solution relies on the level one constraints (velocity level constraints)

- Implications:
  
  - Since the level zero constraints are not enforced, there will be drift in the solution.
  
  - Stabilization terms, that penalize the violation of the level zero constraints, are added to the level one bilateral and unilateral constraints

  - Bilateral and unilateral constraints massaged into the following (superscript \((l)\) denotes the time step):

    \[
    i \in \mathcal{B} : \quad \frac{1}{h} \Psi_i(q^{(l)}(t), t) + \nabla \Psi_i^T \mathbf{v}^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0
    \]

    \[
    i \in \mathcal{A} : \quad 0 \leq \gamma_{i,n} \perp \quad \frac{1}{h} \Phi_i(q^{(l)}) + D_i^T \mathbf{v}^{(l+1)} \geq 0 .
    \]

    * Reminiscent of a Baumgarte stabilization scheme
The Discretization Process

- The discretized form of the DVI problem:

\[
M(v^{(l+1)} - v^{(l)}) = hf(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{i \in B} \gamma_{i,b} \nabla \Psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w})
\]

\[
i \in B : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0
\]

\[
i \in A : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0
\]

\[
(\gamma_{i,u}, \gamma_{i,w}) = \arg\min_{\mu \gamma_{i,n} \geq \sqrt{x^2+y^2}} v^T (x D_{i,u} + y D_{i,w})
\]

\[
q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}.
\]

- The first four of the equations above together combine for an optimization problem with equilibrium constraints.

- Why an optimization problem?
  - Because the way the Coulomb friction model is posed

- What type of optimization problem?
  - This represents a nonlinear optimization problem
  - Can be linearized if the friction cone is discretized and represented as a multifaceted pyramid (problem size increases & anisotropy creeps in)

- What are the ’equilibrium constraints’?
  - Your typical optimization problem might display algebraic equality or inequality constraints
  - Above, we are solving an optimization problem for which the constraints represent the discretization of a set of differential equations
The NCP $\rightarrow$ CCP Metamorphosis

- Dealing with some generic nonlinear optimization problem like the one above is daunting

- Trick used to recast it as a simpler optimization problem for which
  (i) We are guaranteed that a solution exists (ideally, it would be unique, in some sense), and
  (ii) There are tailored algorithms that we can use to efficiently find the solution

- Trick (coming from the left field): introduce a relaxation of the complementarity constraints

  Instead of working with this:

  $$i \in A : 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0$$

  Work with this:

  $$i \in A : 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0$$

- Owing to this relaxation, the NCP problem becomes a cone complementarity problem (CCP)
The Cone Complementarity Problem

- The relaxed problem we have to deal with now looks like this

\[
M(v^{(l+1)} - v^{(l)}) = hf(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{i \in B} \gamma_{i,b} \nabla \psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w})
\]

\[
i \in B : \quad \frac{1}{h} \psi_i(q^{(l)}, t) + \nabla \psi_i^T v^{(l+1)} + \frac{\partial \psi_i}{\partial t} = 0
\]

\[
i \in A : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0
\]

\[
(\gamma_{i,u}, \gamma_{i,w}) = \begin{array}{c}
\arg\min \quad v^T (x D_{i,u} + y D_{i,w})
\end{array}
\]

\[
\sqrt{x^2 + y^2} \leq \mu_i \gamma_{i,n}
\]

\[
q^{(l+1)} = q^{(l)} + hL(q^{(l)}) v^{(l+1)}.
\]
Cone Complementarity Problem (CCP)

- After some algebraic massaging, the equations on the previous slide combine to lead to the following CCP:

  - Introduce the convex hypercone...

    \[ \Upsilon = \left( \bigoplus_{i \in \mathcal{A}(q^{(l)})} \mathcal{FC}^i \right) \bigoplus \left( \bigoplus_{i \in \mathcal{B}(q^{(l)})} \mathcal{BC}^i \right) \]

    where \( \mathcal{FC}^i \) is the \( i \)-th friction cone

    \( \mathcal{BC}^i \) is \( \mathbb{R} \)

  - ... and its polar hypercone

    \[ \Upsilon^\circ = \left( \bigoplus_{i \in \mathcal{A}(q^{(l)})} \mathcal{FC}^{i\circ} \right) \bigoplus \left( \bigoplus_{i \in \mathcal{B}(q^{(l)})} \mathcal{BC}^{i\circ} \right) \]

  - The CCP that needs to be solved at each time step is as follows:

    * Find the Lagrange hyper-multiplier \( \gamma \) that satisfies:

    \[ \Upsilon \ni \gamma \perp -(\mathbf{N}\gamma + \mathbf{r}) \in \Upsilon^\circ \]

    * The matrix \( \mathbf{N} \) and vector \( \mathbf{r} \) are given, computed based on state information at time-step \( t^{(l)} \)
The Optimization Angle

- CCP represents first order optimality condition (KKT conditions) for a quadratic problem with conic constraints

\[
\min_{\gamma} \frac{1}{2} \gamma^T N \gamma + r^T \gamma
\]

subject to \( \gamma_i \in \mathcal{Y}_i \) for \( i = 1, 2, \ldots, N_c \).

- \( N \in \mathbb{R}^{3N_c \times 3N_c} \) is symmetric and positive semi-definite
- \( N \) and \( r \in \mathbb{R}^{3N_c} \) do not depend on \( \gamma \). They are computed once at the beginning of each time step
- The problem is convex, therefore it has a global solution
- Problem does not have a unique solution (since \( N \) is not positive-definite)
Wrapping it Up, Complementarity Approach

- Everything straightforward once frictional contact forces are available

- The velocity $v^{(l+1)}$ is computed via a matrix-vector multiplication

- Once velocity available, generalized positions $q^{(l+1)}$ computed as

$$q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}$$
Complementarity Approach: Putting Things in Perspective

- Perform collision detection

- Formulate equations of motion; i.e., pose DVI problem

- DVI discretized to lead to nonlinear complementarity problem (NCP)

- Relax NCP to get CCP

- Equivalently, solve QP with conic constraints to compute $\gamma$

- Once friction and contact forces available, velocity available

- Once velocity available, positions are available (numerical integration)
Additive Manufacturing (3D SLS Printing)

Courtesy of Professor Tim Osswald, Polymer Engineering Center, UW-Madison
# Selective Laser Sintering (SLS) Layering

## Granular Material

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>N</td>
<td>1,186,185</td>
</tr>
<tr>
<td>(\rho)</td>
<td>930 [kg/m(^3)]</td>
</tr>
<tr>
<td>(r(\text{mean}))</td>
<td>0.029 [mm]</td>
</tr>
<tr>
<td>(r(\sigma))</td>
<td>0.0075 [mm]</td>
</tr>
</tbody>
</table>

## Simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation Length:</td>
<td>20 [s]</td>
</tr>
<tr>
<td>(\Delta t)</td>
<td>(5 \times 10^{-5}) [s]</td>
</tr>
<tr>
<td>Run Time</td>
<td>49 Hours</td>
</tr>
</tbody>
</table>
Dress 3D Printing Problem
Using Simulation in 3D Printing of Clothes
Pros and Cons, Complementarity Approach

• Pros
  • Allows for large integration step sizes since it doesn’t have to deal with contact stiffness
  • Reduced number of model parameters one can fiddle with
  • It looks at the entire problem, it doesn’t artificially decouples the problem

• Cons
  • Requires a global solution, which means that large systems lead to large coupled problems
  • Our implementation has numerical artifacts owing to the relaxation of the non-penetration condition
  • Challenging to model coefficient of restitution (currently uses an inelastic model)
  • Stuck w/ a rigid body dynamics take on the problem (can’t propagate shock waves)
Reference, DVI Literature

• Lab technical report:


Closing Remarks

[Applies both for Penalty and DVI approaches]

• There is some hand waving when it comes to handling friction and contact
  • Both in Penalty and DVI

• Handling frictional contact is equally art and science
  • To get something to run robustly requires tweaking
  • Takes some time to understand strong/weak points of each approach

• Continues to be area of active research
Supplemental Slides
General Comments, DVI

• Differential Variational Inequality (DVI): a set of differential equations that hold in conjunction with a collection of constraints

  • Classical equations of motion: Newton-Euler EOMs, govern time evolutions of constrained MBS

  • Kinematic constraints coming from joints
    • These constraints are called bilateral constraints

  • When dealing with contacts, the non-penetration condition captured as a unilateral constraint
    • At point of contact, relative to body 1, body 2 can move outwards, but not inwards

  • The variational attribute stems from the optimization problem posing the Coulomb friction model
Nomenclature: classical MBD uses kinematic constraints, which we’ll call bilateral constraints. In DVI we also have non-penetration constraints, which are unilateral constraints and assume the form of inequalities.

Notation: We’ll call $\mathcal{A}$ the set of all active unilateral constraints present in the system. Think of these as active contacts. They’ll be denoted by

$$\Phi_i(q) \quad i \in \mathcal{A}$$

— Note that the nonpenetration condition is expressed as (the distance between two bodies should also be positive)

$$\Phi_i(q) \geq 0, \quad i \in \mathcal{A}$$

Notation: We’ll call $\mathcal{B}$ the set of all bilateral constraints present in the system. These expression of these constraints will be denoted by $\Psi(q, t)$. Just like before we have that

$$\Psi_i(q, t) = 0, \quad i \in \mathcal{B}$$

Remark: While the bilateral constraints typically don’t change in time (a spherical joint stays a spherical joint throughout the simulation), the unilateral constraints appear and disappear; i.e., contacts are made and then broken. In other words, $\mathcal{A}$ depends on the state $q$ of the system.