The Absolute Nodal Coordinate Formulation – ANCF

Antonio Recuero
Dan Negrut

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Abstract This white paper describes the fundamentals of the nonlinear finite element theory used to implement ANCF finite elements in Chrono. The finite elements addressed in this document are: the gradient-deficient ANCF beam element and the bi-linear ANCF laminated shell element with orthotropic material properties. This ANCF shell element is also gradient deficient, as it only has one single gradient vector, and implements finite element numerical techniques to avoid kinematic locking, namely the assumed natural strain (ANS) and the enhanced assumed strain (EAS). This finite element's formulation has been devised for tire modeling applications within the context of flexible multibody dynamics and large-scale vehicle simulations.

Keywords: Laminated ANCF shell element, tire dynamics, vehicle mobility, Chrono , open-source code, high-performance computing

Contents

1. Introduction

The absolute nodal coordinate formulation (ANCF) is a nonlinear finite element formulation originated in the field of flexible multibody dynamics to describe large deformation of moving bodies. This formulation was introduced by Shabana [4] and contrasted with the co-rotational or floating frame of reference formulations because no co-rotated frame was used to describe the kinematics of deformed finite elements. Arguably, the most distinguishing feature of ANCF is the use of position vector gradients to describe the rotation of the body as well as its strain state, thereby avoiding the need for interpolating non-vectorial rotation parameters [3]. Without intention of completeness, we present a summary of the method in this section.

ANCF uses nodal, global position and position vector gradient vectors to describe the dynamics of flexible bodies that can experience large (as opposed to small) deformation; some authors identify this deformation magnitude as "moderate", leaving the term "large" for massive deformation, e.g. for Eulerian-like approaches where the material flows. In general, the position field of an ANCF element may be defined as

$$\underbrace{\boldsymbol{r}^i(x,y,z,t)}_{\substack{\text{Position of an arbitrary}\\ \text{point within the element}}} = \underbrace{\boldsymbol{S}(x,y,z)}_{\substack{\text{Space-dependent}\\ \text{shape function}}} \times \underbrace{\boldsymbol{q}^i(t),}_{\substack{\text{Vector of nodal}\\ \text{degrees of freedom}}} \tag{1}$$

where i is an arbitrary finite element, x, y, z are local element parameters, t is the time, $\boldsymbol{S}$ a matrix of shape functions, and $\boldsymbol{q}^j$ is a vector containing nodal coordinates.

One classification for ANCF elements may be based on the number of position vector gradients per node, as follows:

- **Fully parameterized**. These ANCF finite elements possess a full set of gradient vectors, that is, each node has one position vector $\boldsymbol{r}$, and three position vector gradients $\boldsymbol{r}_x$, $\boldsymbol{r}_y$, and $\boldsymbol{r}_z$. Fully parameterized elements can straightforwardly implement continuum mechanics approaches, which usually rely on deformation gradient tensors, $\boldsymbol{F}$.

- **Gradient deficient**. Many ANCF finite elements do not have a full set of gradients for several reasons; e.g. one or two position vector gradients may suffice to define a volume and, therefore, to use a continuum mechanics approach. Further, the use of a class of gradient-deficient elements has been found to successfully eliminate diverse locking problems (poor description of elemental strain state).

- **Higher-order coordinates**. Finite element technology allows the use of higher-order derivatives of position vectors as nodal coordinates. This has found to be beneficial to employ description of strains based on continuum mechanics [5].

There is an extensive body of literature that deal with the development of ANCF finite elements, including beams, shells, plates, and solids. Here, we are only going to summarize the formulation of two ANCF finite elements implemented in Chrono.

2. ANCF Beam Element

The ANCF beam element implemented in Chrono is a gradient-deficient element that was introduced by Berzeri and Shabana [2]. This beam element, sometimes called "cable" element, consists of two nodes which have a position vector and a position vector gradient along the beam center axis as coordinates (see Fig. 1). The coordinates of a node k may be expressed as $\boldsymbol{q}^k(t) = \begin{bmatrix} \boldsymbol{r}^{k\mathrm{T}} & \boldsymbol{r}_x^{k\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. The position field of the ANCF beam element is defined as

$$\boldsymbol{r}^i = \begin{bmatrix} s_1\boldsymbol{I} & s_2\boldsymbol{I} & s_3\boldsymbol{I} & s_4\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}^{1\mathrm{T}} & \boldsymbol{q}^{2\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = \boldsymbol{S}\left(x\right)\boldsymbol{q}^i, \tag{2}$$

where the vector $\boldsymbol{q}^i$ has the coordinates of both nodes, and the shape functions are defined as

$$\begin{aligned} s_1 &= 1 - 2x^2 + 2x^3, \\ s_2 &= l\left(x - 2x^2 + x^3\right), \\ s_3 &= 3x^2 - 2x^3, \\ s_4 &= l\left(-x^2 + x^3\right), \end{aligned} \tag{3}$$

where x is the local parameter of the element ($x = 0$ at the first node) and l is the finite element's reference length.

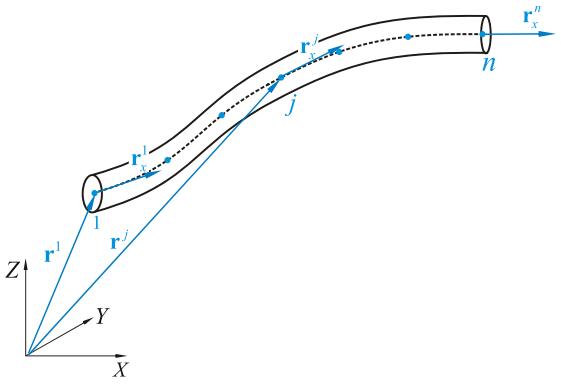

Figure 1: ANCF beam element's kinematic description

Elastic Forces

Two strains fully define the internal forces of this element: the longitudinal stretch, ε_x, and the curvature, κ. The virtual work exerted by the internal forces may be written as follows

$$\delta W_e = \int_L [EA\varepsilon_x \delta\varepsilon_x + EI\kappa\delta\kappa]\mathrm{d}x, \tag{4}$$

where E, A, and I are the modulus of elasticity, the cross section area, and the area moment of inertia, respectively; the longitudinal stretch and curvature are

$$\varepsilon_x = \frac{1}{2}\left(\boldsymbol{r}_x^{\mathrm{T}}\boldsymbol{r}_x - 1\right) \text{ and } \kappa = \frac{|\boldsymbol{r}_x \times \boldsymbol{r}_{xx}|}{|\boldsymbol{r}_x|^3},$$

respectively, where $\boldsymbol{r}_{xx} = \partial^2\boldsymbol{r}/\partial x^2$.

Inertia Forces

The inertia forces take a simple form in ANCF; this is due to the description of bodies' kinematics directly in global coordinates. The velocity of any point within an element i may be written as

$$\underbrace{\dot{\boldsymbol{r}}^i(x,y,z,t)}_{\substack{\text{Velocity of an arbitrary}\\ \text{point within the element}}} = \underbrace{\boldsymbol{S}(x,y,z)}_{\substack{\text{Space-dependent}\\ \text{shape function}}} \times \underbrace{\dot{\boldsymbol{q}}^i(t).}_{\substack{\text{Vector of}\\ \text{generalized velocities}}} \tag{5}$$

The kinetic energy of a finite element i may be obtained as

$$T = \frac{1}{2}\int_V \rho\dot{\mathbf{r}}^{i\mathrm{T}}\dot{\mathbf{r}}^i \,\mathrm{d}V = \frac{1}{2}\dot{\mathbf{q}}^{i\mathrm{T}}\mathbf{M}\dot{\mathbf{q}}^i. \tag{6}$$

The mass matrix is defined as $\mathbf{M} = \int_A \rho A\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S}\,\mathrm{d}x = \text{ constant}$. By analyzing Eqs. (2) and (6), the reader may realize that the computation of inertia forces is much simpler than that of internal forces. This observation is valid not only for the beam element presented in this section, but for all ANCF finite elements: *Mass matrix is constant but the internal forces depend heavily nonlinearly on the coordinates.* It may also be noted that this particular ANCF beam element assumes that the section is axisymmetric, that is, the cross section properties are assumed to be same along any axis.

The Jacobian of the elastic forces is needed for the solution of index-3 DAEs in Chrono using implicit numerical integrators. Currently, this Jacobian is computed numerically via finite differences.

3. Chrono's ANCF Shell Element

Chrono's users may find detailed descriptions of the Chrono's laminated ANCF implementation in Refs.[6, 7].

The basic kinematics of the absolute nodal coordinate formulation (ANCF) shell finite element implemented in Chrono is depicted in Fig.2. The nodal position is defined as a function of the global position and the transverse gradient vector $\mathbf{r}_z^i = \frac{\partial \mathbf{r}^i}{\partial z^i}(x^i, y^i)$ which describes the orientation of the cross section. Element i's positions and gradients on the mid-plane can be fully described as

$$\mathbf{r}_m^i(x^i, y^i) = \mathbf{S}_m^i(x^i, y^i)\mathbf{e}_p^i, \quad \frac{\partial \mathbf{r}^i}{\partial z^i}(x^i, y^i) = \mathbf{S}_m^i(x^i, y^i)\mathbf{e}_g^i, \tag{7}$$

where x^i and y^i refer to element i's local coordinates in the parametric space, $\mathbf{S}_m^i = [S_1^i\mathbf{I}\ \ S_2^i\mathbf{I}\ \ S_3^i\mathbf{I}\ \ S_4^i\mathbf{I}]$ is a bilinear shape function matrix, $\mathbf{e}_p^{ik} = \mathbf{r}^{ik}$ is the position vector of node k of element i, and $\mathbf{e}_g^{ik} = \partial \mathbf{r}^{ik}/\partial z^i$ is the position vector gradient of node k of element i (current and reference coordinates –taken from the initial configuration– are stored in the ANCF shell element. The bilinear shape functions of the ANCF shell element are given by the following expressions

$$S_1^i = \frac{1}{4}(1-\xi^i)(1-\eta^i), S_2^i = \frac{1}{4}(1+\xi^i)(1-\eta^i),$$
$$S_3^i = \frac{1}{4}(1+\xi^i)(1+\eta^i), \text{ and } S_4^i = \frac{1}{4}(1-\xi^i)(1+\eta^i).$$

Note that shape functions, position vector gradients, angles, transformation matrices, intermediate operations between frames of reference, and strains are adimensional. The position of an arbitrary point in the shell may be described as

$$\mathbf{r}^i(x^i, y^i, z^i) = \mathbf{S}^i(x^i, y^i, z^i)\mathbf{e}^i, \tag{8}$$

where the combined shape function matrix is given by $\mathbf{S}^i = [\mathbf{S}_m^i\ \ z^i\mathbf{S}_m^i]$. Similarly, the coordinates of the element may be grouped together as $\mathbf{e}^i = [(\mathbf{e}_p^i)^T\ \ (\mathbf{e}_g^i)^T]^T$. Note that Eq. (8) incorporates the element parametric coordinate along the element thickness z^i. Relying on this kinematic description of the shell element, the Green-Lagrange strain tensor may be calculated as

$$\mathbf{E}^i = \frac{1}{2}\left((\mathbf{F}^i)^T\mathbf{F}^i - \mathbf{I}\right), \tag{9}$$

where $\mathbf{F}^i$ is the deformation gradient matrix defined as the current configuration over the reference configuration. Using the current absolute nodal coordinates, this matrix may be defined as

$$\mathbf{F}^i = \frac{\partial \mathbf{r}^i}{\partial \mathbf{X}^i} = \frac{\partial \mathbf{r}^i}{\partial \mathbf{x}^i}\left(\frac{\partial \mathbf{X}^i}{\partial \mathbf{x}^i}\right)^{-1} \tag{10}$$

The strain tensor can then expressed in vector form in the following manner

$$\boldsymbol{\varepsilon}^i = \begin{bmatrix}\varepsilon_{xx}^i & \varepsilon_{yy}^i & \gamma_{xy}^i & \varepsilon_{zz}^i & \gamma_{xz}^i & \gamma_{yz}^i\end{bmatrix}^T \tag{11}$$

where $\boldsymbol{\varepsilon}^i$ is the engineering strain vector in the deformed configuration, computed in Chrono API. Strain derivatives are calculated in order to obtain generalized forces and an internal damping contribution. The elastic internal forces

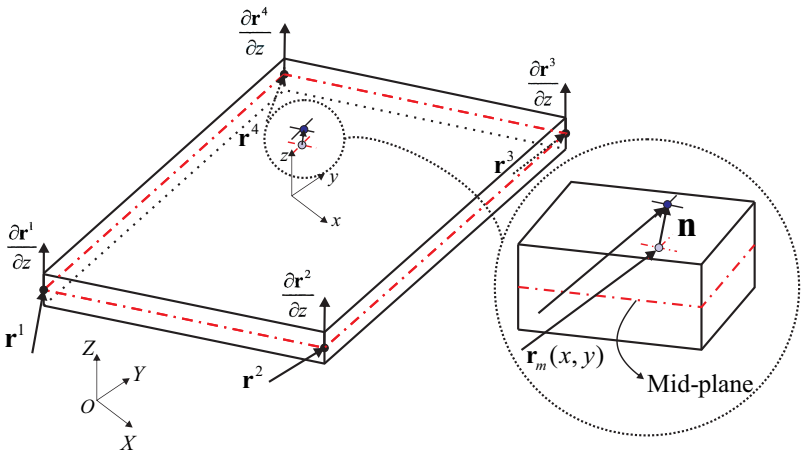

Figure 2: ANCF shell element's kinematic description

are spatially integrated over the element volume using Gaussian quadrature:

$$\mathbf{Q}_k^i = -\int_{V_0} \left(\frac{\partial \varepsilon^c}{\partial \mathbf{e}^i}\right) \frac{\partial W^i(\varepsilon^c + \varepsilon^{EAS})}{\partial \varepsilon^i} \mathrm{d}V_0 \tag{12}$$

where ε^c is the compatible strain, obtained from the displacement field using "Assumed Natural Strain" interpolation to avoid transver/in-plane shear. Further, the term $W^i(\varepsilon^c + \varepsilon^{EAS})$ denotes the strain energy density function, which must be obtained by adding an enhanced strain contribution, ε^{EAS}. The second Piola–Kirchhoff stress tensor is obtained from the relation $\sigma^i = \frac{\partial W^i(\varepsilon^c + \varepsilon^{EAS})}{\partial \varepsilon^i}$. The addition of assumed natural strains and enhanced strains finds justifications of the mixed variational principle by Hu–Washizu [1].

3.1 Computation of Strains for Curved, Orthotropic Shells

Chrono allows the user to create initial geometries that will be automatically considered as "reference" by using covariant transformations. This is internally carried out in the implementation of the element. A frame of reference in the initial (reference) configuration is written as

$$(\mathbf{g}_0)_1 = (\mathbf{r}_0)_x = \frac{\partial \mathbf{r}_0}{\partial x}, \quad (\mathbf{g}_0)_2 = (\mathbf{r}_0)_y = \frac{\partial \mathbf{r}_0}{\partial y}, \quad (\mathbf{n}_0) = (\mathbf{r}_0)_z = \frac{\partial \mathbf{r}_0}{\partial z}. \tag{13}$$

where subscript 0 denotes initial. The unit base vectors of the local Cartesian frame may be obtained as

$$(\mathbf{e}_0)_1 = \frac{(\mathbf{g}_0)_1}{|(\mathbf{g}_0)_1|}, \quad (\mathbf{e}_0)_3 = \mathbf{n}_0, \quad (\mathbf{e}_0)_2 = (\mathbf{e}_0)_3 \times (\mathbf{e}_0)_1. \tag{14}$$

For orthotropic materials, the mechanical behavior depends on fiber orientations. For this reason, it is necessary to include fiber angle in the definition of the local Cartesian frame. Assuming that θ represents the fiber angle with respect to the X axis of the previously calculated local Cartesian frame, the new basis takes the following form:

$$\begin{aligned} (\mathbf{e}_0)_1^{Or} &= (\mathbf{e}_0)_1 \cos\theta + (\mathbf{e}_0)_2 \sin\theta, \\ (\mathbf{e}_0)_2^{Or} &= -(\mathbf{e}_0)_1 \sin\theta + (\mathbf{e}_0)_2 \cos\theta, \quad (\mathbf{e}_0)_3^{Or} = (\mathbf{e}_0)_3. \end{aligned} \tag{15}$$

The relation between the two covariant base vectors may be expressed as

$$(\mathbf{g}_0)_{i'} = \beta_{i'}^{j} (\mathbf{e}_0)_{1,j}^{Or}, \tag{16}$$

where scalars $\beta_{i'}^{j}$ are obtained via a dot product between two adimensional vectors. In matrix form, the coefficients of contravariance transformation may be obtained from the Jacobian of the position vectors at the reference configuration and the local Cartesian frame including anisotropy in the following form

$$\boldsymbol{\beta} = \begin{bmatrix} \mathbf{Y}^{-1}|_{C1}{}^{\mathrm{T}} \\ \mathbf{Y}^{-1}|_{C2}{}^{\mathrm{T}} \\ \mathbf{Y}^{-1}|_{C3}{}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} (\mathbf{e}_0)_1 & (\mathbf{e}_0)_2 & (\mathbf{e}_0)_3 \end{bmatrix} \tag{17}$$

where $\mathbf{Y}^{-1}|_{Ci}$ is the i column of the inverse of $\mathbf{Y} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \begin{bmatrix} (\mathbf{g}_0)_1 & (\mathbf{g}_0)_2 & \mathbf{n}_0 \end{bmatrix}$. The components of the 3-by-3 matrix $\boldsymbol{\beta}$ are used to set up a transformation matrix necessary for the calculation of strains:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \tag{18}$$

where $\beta_{ij} = \beta(i,j)$. Finally the compatible strains are calculated as:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\beta^{\mathrm{T}} \left(\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} - \begin{bmatrix} (g_0)_{11} & (g_0)_{12} & (g_0)_{13} \\ (g_0)_{21} & (g_0)_{22} & (g_0)_{23} \\ (g_0)_{31} & (g_0)_{32} & (g_0)_{33} \end{bmatrix} \right) \beta = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \tag{19}$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $(g_0)_{ij} = (\mathbf{g}_0)_i \cdot (\mathbf{g}_0)_j$.

3.2 ANCF Shell Element Implementation

3.2.1 Initial Steps

Chrono calls the method `MyForce::Evaluate` to evaluate the internal forces of the shell element at one Gauss point. The orientations and contravariant vectors are computed in order to perform calculation of strains as orthotropic material which may an initially curved configuration.

3.2.2 Locking Remedies - Enhanced Assumed Strain

The Enhanced Assumed Strain (EAS) is added to the compatible strain in order to alleviate transverse shear locking. From Eq. (10), we can define the following matrix $\mathbf{J}^i$, which relates the initial and reference configuration:

$$\mathbf{J}^i = \frac{\partial \mathbf{X}^i}{\partial \mathbf{x}^i}, \tag{20}$$

where $\mathbf{J}^i$ is the inverse of the deformation gradient and is used to construct a constant transformation matrix (note that $\mathbf{J}^i$ only depends on the initial and reference configuration).

$$\mathbf{T}^i = \begin{bmatrix} (J^i_{11})^2 & (J^i_{12})^2 & 2J^i_{11}J^i_{12} & (J^i_{13})^2 & 2J^i_{11}J^i_{13} & 2J^i_{12}J^i_{13} \\ (J^i_{21})^2 & (J^i_{22})^2 & 2J^i_{21}J^i_{22} & (J^i_{23})^2 & 2J^i_{21}J^i_{23} & 2J^i_{22}J^i_{23} \\ J^i_{11}J^i_{21} & J^i_{12}J^i_{22} & J^i_{11}J^i_{22}+J^i_{12}J^i_{21} & J^i_{13}J^i_{23} & J^i_{11}J^i_{23}+J^i_{13}J^i_{21} & J^i_{12}J^i_{23}+J^i_{13}J^i_{22} \\ (J^i_{31})^2 & (J^i_{32})^2 & 2J^i_{31}J^i_{32} & (J^i_{33})^2 & 2J^i_{31}J^i_{33} & 2J^i_{32}J^i_{33} \\ J^i_{11}J^i_{31} & J^i_{12}J^i_{32} & J^i_{11}J^i_{32}+J^i_{12}J^i_{31} & J^i_{13}J^i_{33} & J^i_{11}J^i_{33}+J^i_{13}J^i_{31} & J^i_{12}J^i_{33}+J^i_{13}J^i_{32} \\ J^i_{21}J^i_{31} & J^i_{22}J^i_{32} & J^i_{21}J^i_{32}+J^i_{22}J^i_{31} & J^i_{23}J^i_{33} & J^i_{21}J^i_{33}+J^i_{23}J^i_{31} & J^i_{22}J^i_{33}+J^i_{23}J^i_{32} \end{bmatrix} \tag{21}$$

The interpolation matrix for the distribution of the in-plane strains is defined as

$$\mathbf{N}(\xi) = \begin{bmatrix} \xi & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & \xi & \eta & 0 \\ 0 & 0 & 0 & 0 & \zeta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

This matrix guarantees that $\int \mathbf{N}(\xi)\mathrm{d}\xi = 0$. The following 6-by-5 matrix is defined to include the enhanced assumed strain in the internal forces:

$$\mathbf{G}(\xi) = \frac{|\mathbf{J}_0|}{|\mathbf{J}(\xi)|}\mathbf{T}_0^{-T}\mathbf{N}(\xi), \tag{23}$$

where $\mathbf{T}_0$ is a constant transformation matrix obtained by evaluating $\mathbf{T}$ at the center of the element in the reference configuration. With the aid of a vector of internal parameters α, the EAS may be calculated as

$$\boldsymbol{\varepsilon}^{EAS}(\xi) = \mathbf{G}(\xi)\boldsymbol{\alpha}. \tag{24}$$

The total strain will be the addition of the compatible strain and the EAS, as follows

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^c + \boldsymbol{\varepsilon}^{EAS} \tag{25}$$

The EAS strain is a function of the $\boldsymbol{G}$ matrix at each Gaussian integration point and the vector of internal parameters $\boldsymbol{\alpha}$, which is calculated by solving iteratively the equation $\mathbf{h}^i(\mathbf{e}^i, \alpha^i) = \int_{V^i_o} (\frac{\partial \boldsymbol{\varepsilon}^{EAS}}{\partial \alpha^i})^T \frac{\partial \mathbf{W}^i(\boldsymbol{\varepsilon}^c + \boldsymbol{\varepsilon}^{EAS})}{\partial \boldsymbol{\varepsilon}^i} \mathrm{d}V^i_0 = \mathbf{0}$. The values of $\mathbf{h}^i$, $\frac{\partial \mathbf{h}_n}{\partial \alpha_n}$, and $\frac{\partial \mathbf{h}_n}{\partial \alpha_n}$ are computed within the Gaussian integration loop. The updated value of $\boldsymbol{\alpha}$ in the current iteration is defined by $\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n - (\frac{\partial \mathbf{h}_n}{\partial \boldsymbol{\alpha}_n})^{-1}(\mathbf{h}_n + \frac{\partial \mathbf{h}_n}{\partial \mathbf{e}_n}\Delta \mathbf{e}_{n+1})$.

3.2.3 Locking Remedies - Assumed Natural Strain

Assumed natural strains (ANS) are introduced in Chrono's implementation to avoid shear and thickness locking in the shell finite element. Compatible thickness and shear strains, which are interpolated in the computation of internal forces. Additional sampling points A, B, C, and D, located at the middle of the element edges, are used to calculate the ANS strain as follows

$$\varepsilon_{zz}^{ANS} = S_1^{ANS}\varepsilon_{zz}^1 + S_2^{ANS}\varepsilon_{zz}^2 + S_3^{ANS}\varepsilon_{zz}^3 + S_4^{ANS}\varepsilon_{zz}^4 \tag{26}$$

$$\tilde{\gamma}_{xz}^{ANS} = \frac{1}{2}(1-\eta)\,\tilde{\gamma}_{xz}^{C} + \frac{1}{2}(1+\eta)\,\tilde{\gamma}_{xz}^{D} \tag{27}$$

$$\tilde{\gamma}_{yz}^{ANS} = \frac{1}{2}(1-\xi)\,\tilde{\gamma}_{yz}^{A} + \frac{1}{2}(1+\xi)\,\tilde{\gamma}_{xz}^{B}, \tag{28}$$

where tildes denote covariant quantities.

3.2.4 Equations of Motion

The mass matrix of the element is given by

$$\mathbf{M}^i = \int_{V_o^i} \rho_0^i (\mathbf{S}^i)^T \mathbf{S}^i \mathrm{d}V_o^i, \tag{29}$$

which remains constant throughout the simulation. The equations of motion may be written as

$$\mathbf{M}^i \ddot{\mathbf{e}}^i = \mathbf{Q}_k^i(\mathbf{e}^i, \dot{\mathbf{e}}^i, \alpha^i) + \mathbf{Q}_e^i(\mathbf{e}^i, \dot{\mathbf{e}}^i, t), \tag{30}$$

where $\mathbf{Q}_k$ is the element elastic force vector and $\mathbf{Q}_e$ is the external force vector. The Newton differences for $\mathbf{e}_{n+1}$ and α_{n+1} are calculated by solving the following system of equations

$$\begin{bmatrix} \partial\mathbf{f}/\partial\mathbf{e}_n & \partial\mathbf{f}/\partial\alpha_n \\ \partial\mathbf{h}/\partial\mathbf{e}_n & \partial\mathbf{h}/\partial\alpha_n \end{bmatrix} \begin{bmatrix} \Delta\mathbf{e}_{n+1} \\ \Delta\alpha_{n+1} \end{bmatrix} = - \begin{bmatrix} \mathbf{f}_n \\ \mathbf{h}_n \end{bmatrix}. \tag{31}$$

After eliminating $\Delta\alpha_{n+1}$ the following equation can solve for $\Delta\mathbf{e}_{n+1}$ using the following equation:

$$\left(\frac{\partial\mathbf{f}_n}{\partial\mathbf{e}_n} - \frac{\partial\mathbf{f}_n}{\partial\alpha_n} \left(\frac{\partial\mathbf{h}_n}{\partial\alpha_n} \right)^{-1} \frac{\partial\mathbf{h}_n}{\partial\mathbf{e}_n} \right) \Delta\mathbf{e}_{n+1} = -\mathbf{f}_n + \frac{\partial\mathbf{f}_n}{\partial\alpha_n} \left(\frac{\partial\mathbf{h}_n}{\partial\alpha_n} \right)^{-1} \mathbf{h}_n. \tag{32}$$

3.3 Computation of the Jacobian of the Elastic Forces

The Jacobian of the elastic forces are calculated in `MyJacobian::Evaluate`. The method for Jacobian of elastic forces recalculates many of the quantities involved in the computation of internal forces. This is done this way in order to effectively separate the both calculations and allow for single computation of

the Jacobian in each time step (in the future). Basic quantities related to the ANCF shell element internal forces are recalculated for the calculation of the Jacobian.

The Jacobian of the elastic forces is split into different parts:

Jacobian of elastic forces Direct derivation of generalized elastic forces w.r.t. coordinates (Eq. (12)).

Jacobian of EAS forces The Jacobian of the generalized forces coming from the EAS formulation.

The computation of the Jacobian of the elastic forces is called for each layer of material. The total Jacobian, for all the layers of the element, is accumulated if the element features more than one layer.

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