Chrono::FEA

ANCF Theoretical Background

1. ANCF introduction

Why Chrono::FEA?

- Finite elements are necessary for the analysis of dynamical systems
- Mechanical components deform in addition to translating and rotating
- Flexible bodies interact dynamically with other physical components though contact forces, force elements, constraints, etc.

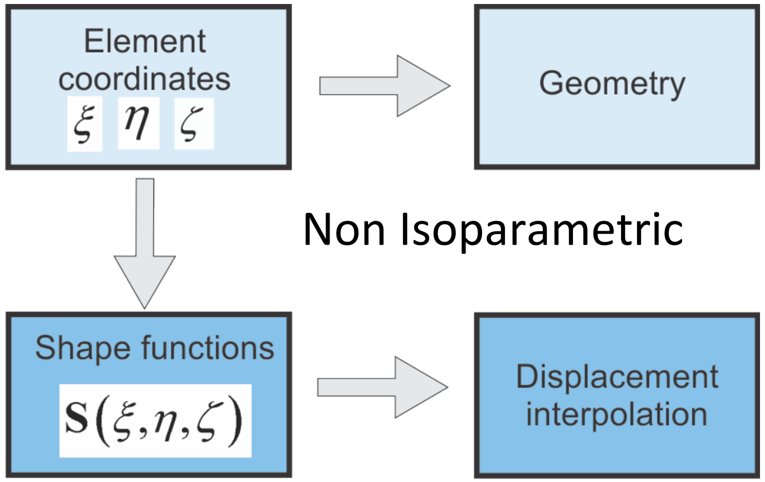
- Chrono::FEA features 3 main formulations for describing flexible bodies or solids:
  - The Absolute Nodal Coordinate Formulation (ANCF)– Large deformation (arbitrary rotation and translation)
  - The Co-rotational Formulation –Small deformation (arbitrary rotation and translation)
  - Traditional Lagrangian Finite Elements –Good for large deformation (automatically includes arbitrary rotation and translation)

Isoparametric finite elements

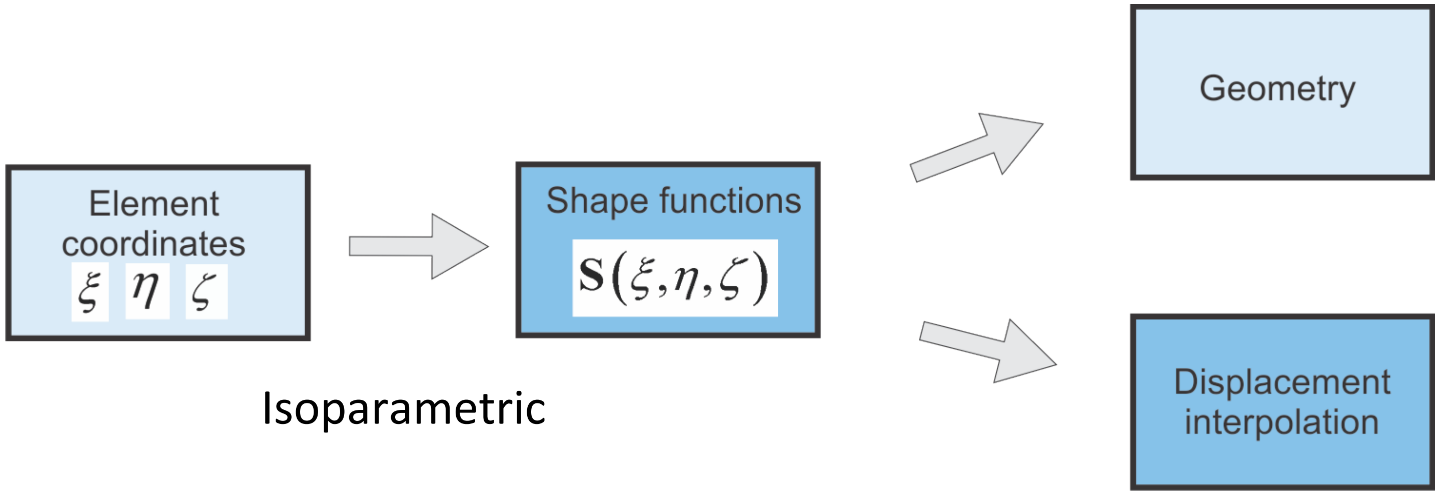
- Free online material: “The Isoparametric Representation”, Chapter 16, C. Felippa
- In isoparametric finite elements, geometry and displacement fields are given by the same parametric representation
  - Note: In FE/FFR, beam/shell finite elements are NOT isoparametric –infinitesimal angles. Shape functions describe displacements, but not geometry.
- The basic principle of isoparametric elements is that the interpolation functions for the displacements are also used to represent the geometry of the element.
- Isoparametric formulation makes it straightforward to have non-rectangular, curved elements

Isoparametric finite elements

- Traditional isoparametric elements only have position coordinates
- Displacements are expressed in terms of the natural (local) coordinates and then differentiated with respect to global coordinates. Accordingly, a transformation matrix [J], called Jacobian, is produced.

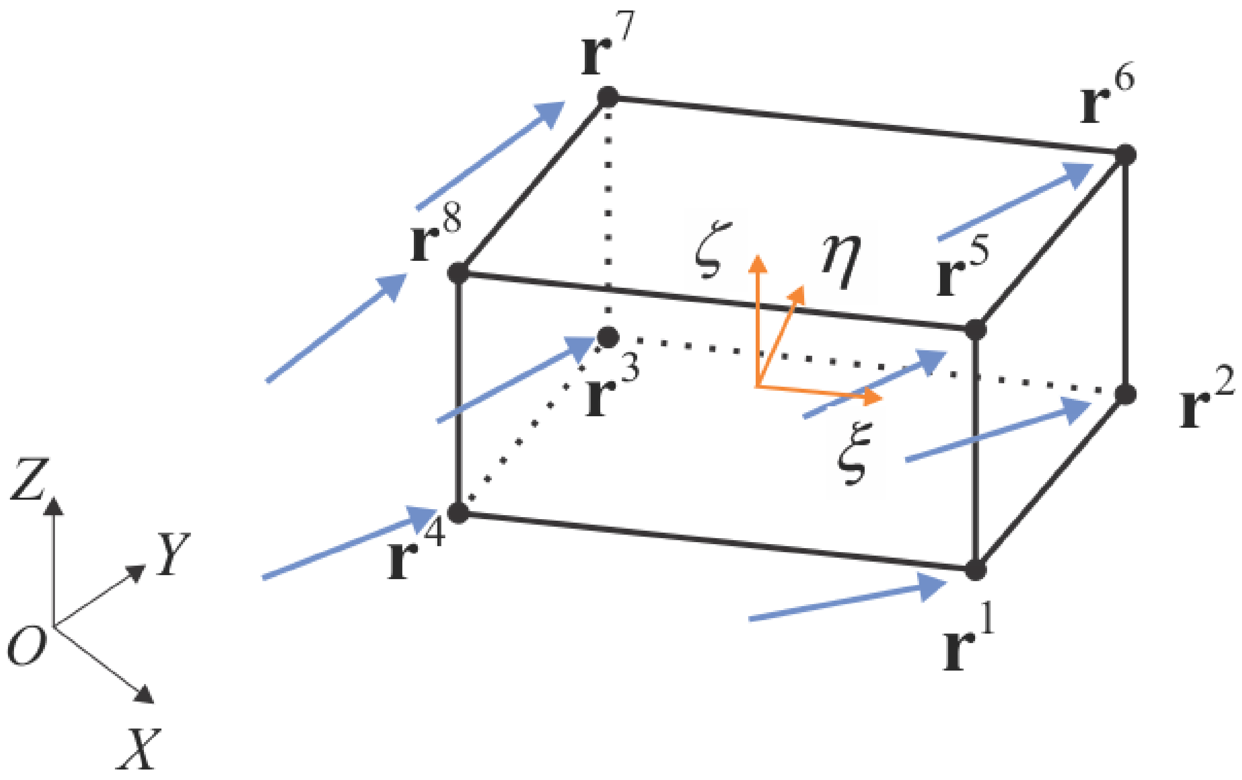


Non Isoparametric



Isoparametric

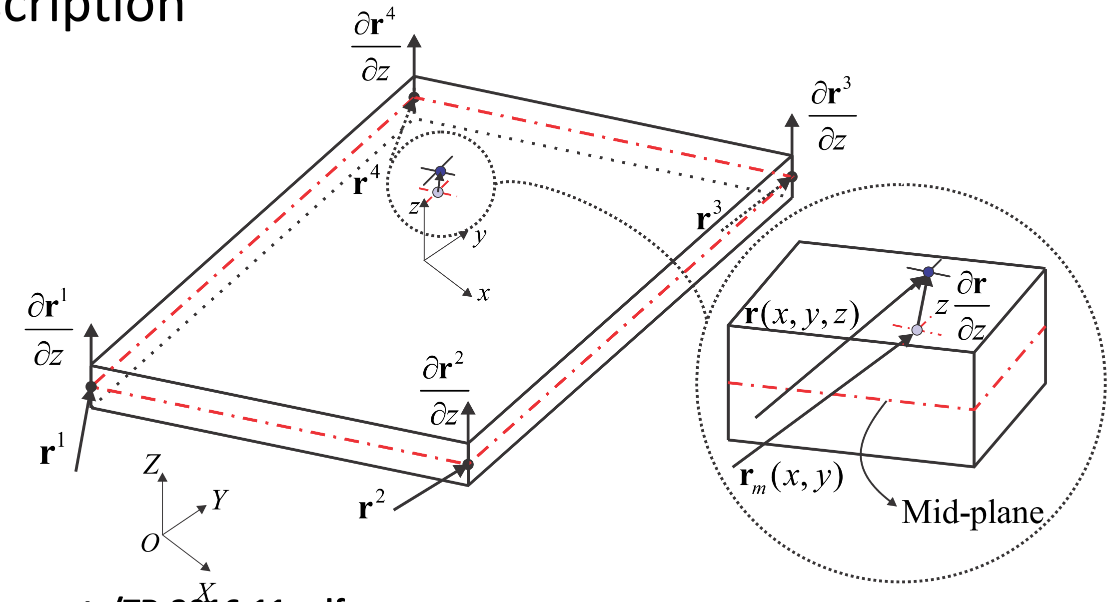
Isoparametric finite elements



$$\mathbf{r} = \mathbf{S}(\xi, \eta, \zeta)\mathbf{e}$$

Absolute nodal coordinate formulation

- ANCF was introduced in 1996 by Prof. Shabana of U. of Illinois
- Uses isoparametric elements with the addition of position vector gradients. That is, nodal coordinates are position vectors and its gradient vectors.
- Position vector gradients univocally define nodal rotation and do not lead to a redundancy problem: Position and rotations being interpolated independently
- Note that the defining feature of this method is the use of extensible gradient vectors, which is built upon a kinematic description



Free document on ANCF implementation in Chrono: **http://sbel.wisc.edu/documents/TR-2016-11.pdf**

Absolute nodal coordinate formulation

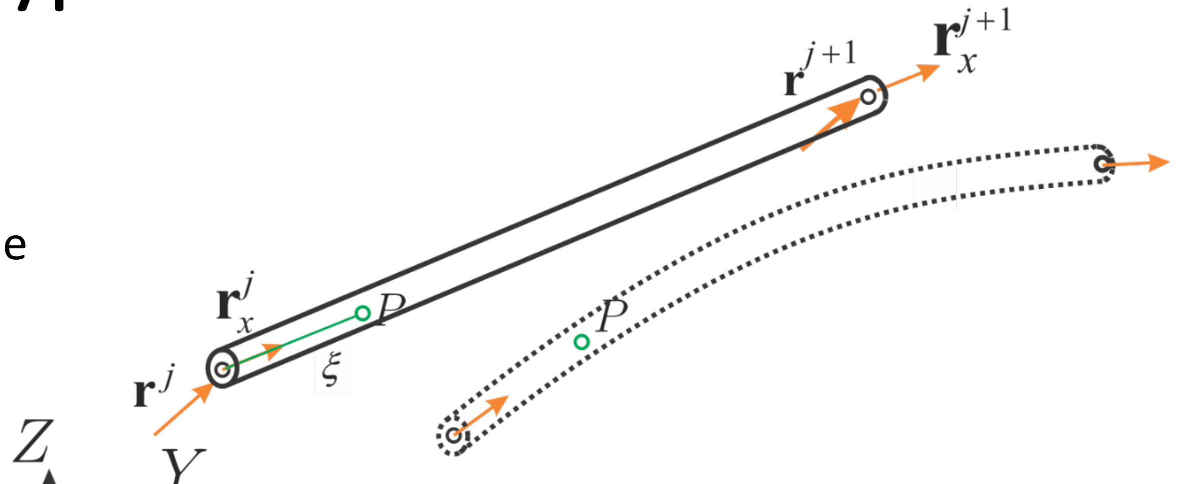
- Only vectorial quantities are interpolated
- Fully non-incremental method: Small increments assumption not needed
- Leads to a constant mass matrix and a “complex’’ definition of material forces
- ANCF elements can be structural-based (beams, shells) or solid-based (bricks, tetrahedra)
- Since ANCF does not impose any kinematic description, special care is put for the development of beam/shell element

ANCF beam elements types

- “Gradient-deficient”
  - Do not necessarily define a volume
  - Structural strains

$$\kappa_m = \frac{|\mathbf{r}_x \times \mathbf{r}_{xx}|}{|\mathbf{r}_x|^2}, \quad \varepsilon_{xx} = \frac{1}{2}\left(\mathbf{r}_x^{\mathrm{T}}\mathbf{r}_x - 1\right)$$

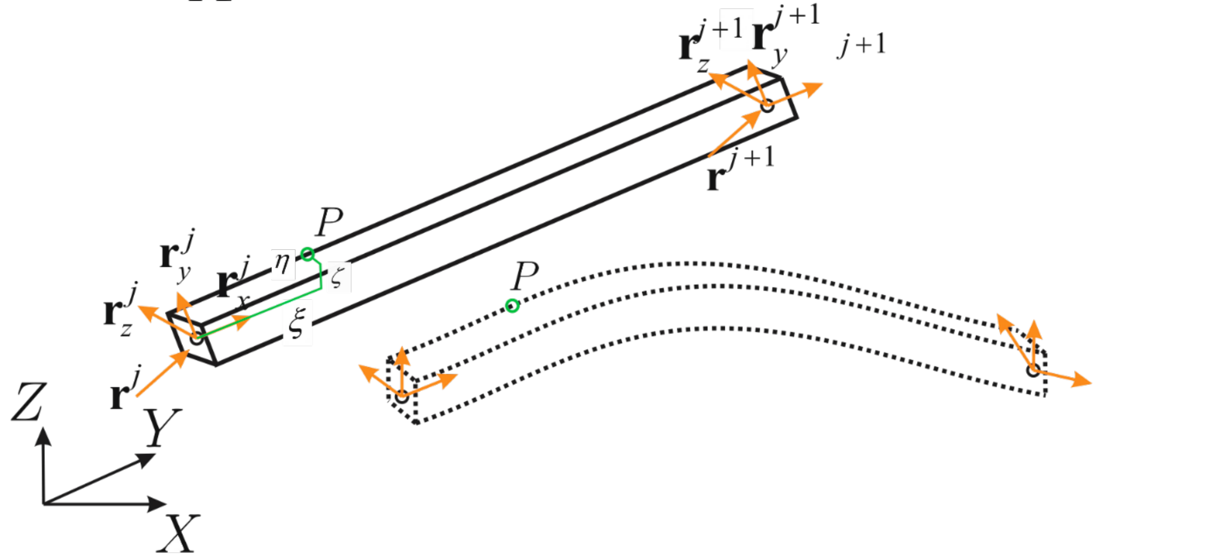
  - No torsion or shear



$$\mathbf{r}^P(\xi) = \mathbf{S}(\xi)\left[\mathbf{r}^j \quad \mathbf{r}_x^j \quad \mathbf{r}^{j+1} \quad \mathbf{r}_x^{j+1}\right]^{\mathrm{T}}$$

- “Fully-parameterized”
  - Full set of gradients
  - Deformation gradient tensor

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{X}} = \left[\mathbf{r}_x^P \quad \mathbf{r}_y^P \quad \mathbf{r}_z^P\right]$$



$$\mathbf{r}^P(\xi,\eta,\zeta) = \mathbf{S}(\xi,\eta,\zeta)\left[\mathbf{r}^j \quad \mathbf{r}_x^j \quad \mathbf{r}_y^j \quad \mathbf{r}_z^j \quad \mathbf{r}^{j+1} \quad \mathbf{r}_x^{j+1} \quad \mathbf{r}_y^{j+1} \quad \mathbf{r}_z^{j+1}\right]^{\mathrm{T}}$$

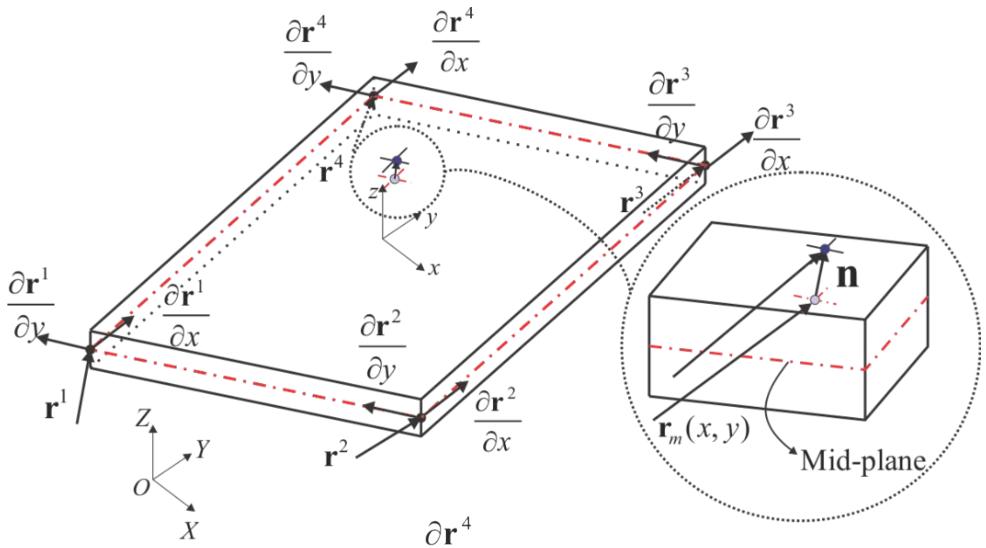
ANCF beam element formulation types

- **Structural Approach**: Geometries of the solid is used to simplify strains and/or make strain definition more precise. This usually involves that there is one or two dimensions of the solid smaller than the other (s)

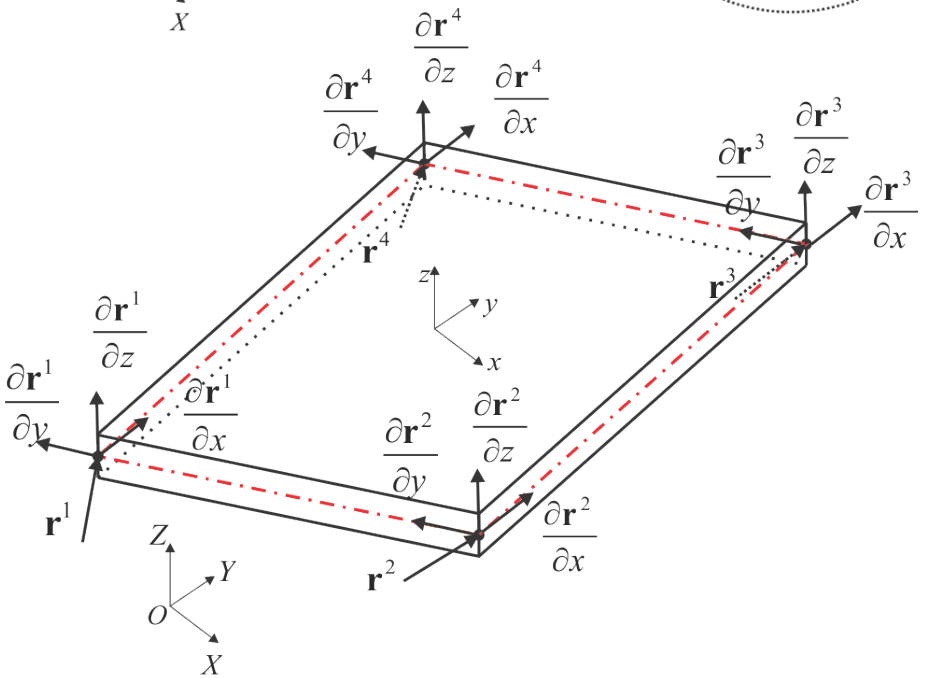
- **Continuum-Based Approach**: It involves strain definitions based on a fully 3D approach: Deformation gradient **F**.

ANCF shell element types

- “Gradient-deficient”
  - For structural approaches
  - Contains in-plane gradient vectors
  - Avoids some types of locking



- “Fully-parameterized”
  - Indicated for continuum-based approaches
  - One of the original elements
  - Severe locking



Constraints with ANCF finite elements

- Any flexible body can be constrained to any other rigid/flexible by using constraints
- This is useful to build general-purpose mechanical systems; e.g. robotic arms.
- Material points and directions are defined in the flexible body

- Example. Spherical joint between two ANCF beam nodes
  - Node j of beam 1
  - Node i of beam 2

$$\boxed{\mathbf{r}^i = \mathbf{r}^j}$$

Constraints to rigid bodies

- Position and direction constraints to rigid bodies, *aka* fixed
- Position; node i, body j

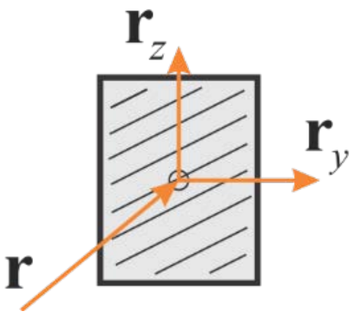
$$\mathbf{r}^i = \mathbf{R}^j + \mathbf{A}^j \bar{\mathbf{u}}^P$$

- Direction:

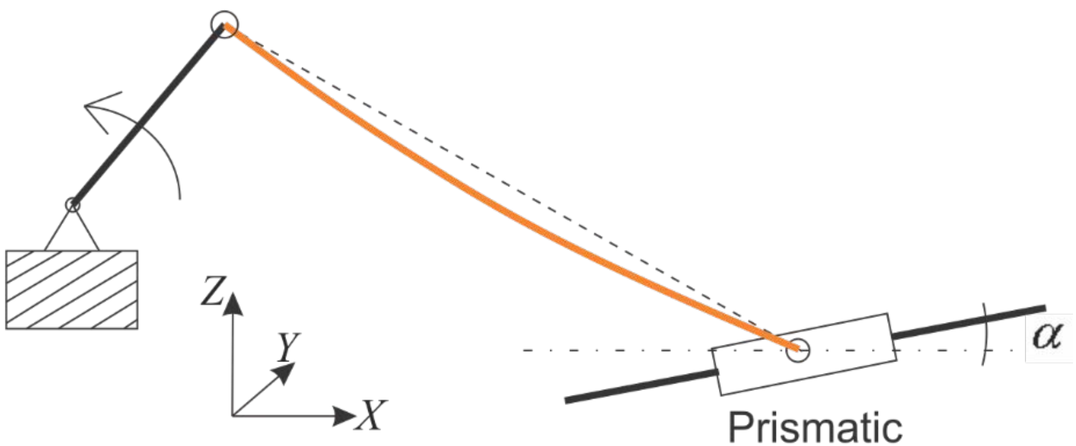
$$\left|\mathbf{r}_x^i\right| = \mathbf{A}^j \bar{\mathbf{d}}^P$$

Constraints to rigid bodies

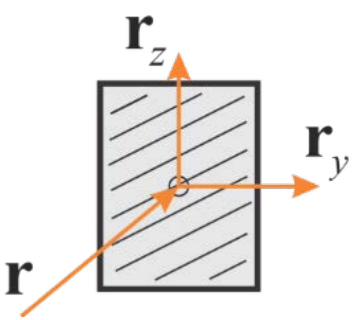
- Example. Crankshaft mechanism with ANCF beam
  - Beam has the following coordinates
    - Position vector (3D)
    - $\mathbf{r}_y$ cross section plane
    - $\mathbf{r}_z$ cross section plane
    - $\mathbf{r}_y$ perpendicular to $\mathbf{r}_z$ in the **undeformed configuration**



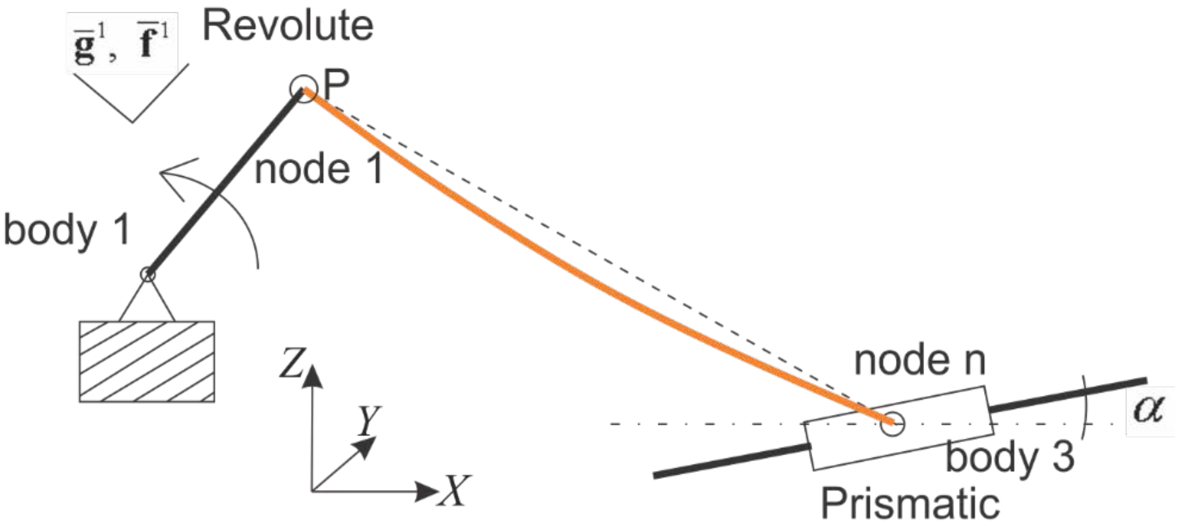
Beam cross section



Constraints to rigid bodies



Beam cross section



Revolute

- Position

$$\mathbf{C}(\mathbf{R}_b^1, \boldsymbol{\theta}_b^1, \mathbf{r}^1, \mathbf{r}_y^1, \mathbf{r}_z^1) = \mathbf{R}_b^1 + \mathbf{A}\left(\boldsymbol{\theta}_b^1\right)\bar{\mathbf{u}}^P - \mathbf{r}^1 = \mathbf{0}$$

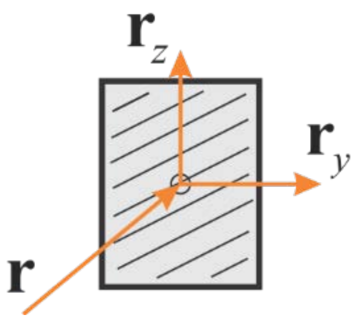
- Local vector in flexible body is parallel to local vector in rigid body: Their global direction define the mechanical joint's axis of revolution

$$\mathrm{C}(\boldsymbol{\theta}_b^1, \mathbf{r}_y^1) = \left(\mathbf{A}^1\left(\boldsymbol{\theta}_b^1\right)\bar{\mathbf{f}}^1\right)^{\mathrm{T}} \mathbf{r}_y^1 = \mathbf{0}$$

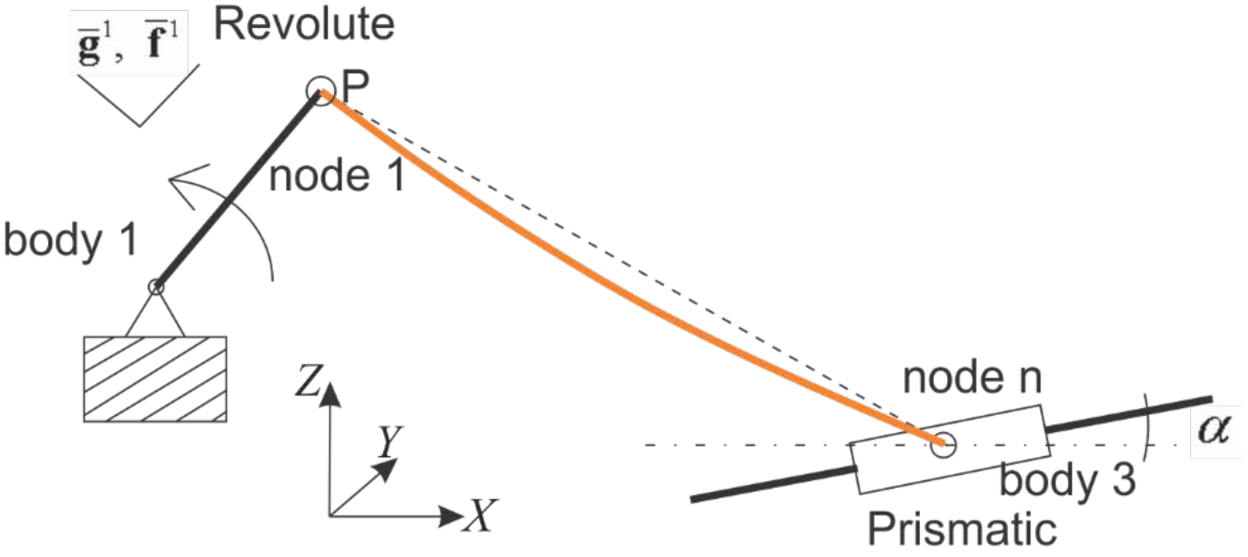
$$\mathrm{C}(\boldsymbol{\theta}_b^1, \mathbf{r}_y^1) = \left(\mathbf{A}^1\left(\boldsymbol{\theta}_b^1\right)\bar{\mathbf{g}}^1\right)^{\mathrm{T}} \mathbf{r}_y^1 = \mathbf{0}$$

- Where $\bar{\mathbf{g}}^1$, $\bar{\mathbf{f}}^1$ are local vectors of body 1 perpendicular to joint axis

Constraints to rigid bodies



Beam cross section



- Similar equations for prismatic/universal/… joints
- Jacobian of these constraint equations can be obtained in a similar manner to regular rigid body constraints
- Jacobian may involve coordinates from rigid bodies, one type of ANCF finite element, another type of ANCF element, etc.

Constraints to rigid bodies

- Constraint relationships and their derivatives can be obtained also for flexible body/flexible body constraints
- Key to defining these joints is
  - Identify what flexible body coordinates represent. E.g. Do they define a fiber orientation? Do they contain a vector perpendicular to the beam cross section?
- Constraints to ANCF bodies also define boundary conditions. These constraints are not straightforward to come up with because of the relations between them and the strains, which often define the boundary conditions.

2. ANCF beams

ANCF: Kinematic definition and kinetic energy

In general, the position of an ANCF element may be defined as

$$\underbrace{\mathbf{r}(x,y,z,t)}_{\substack{\text{Position of an arbitrary}\\ \text{point within the element}}} = \underbrace{\mathbf{S}(x,y,z)}_{\substack{\text{Space-dependent}\\ \text{shape function}}} \underbrace{\mathbf{q}(t)}_{\substack{\text{Vector of nodal}\\ \text{degrees of freedom}}}$$

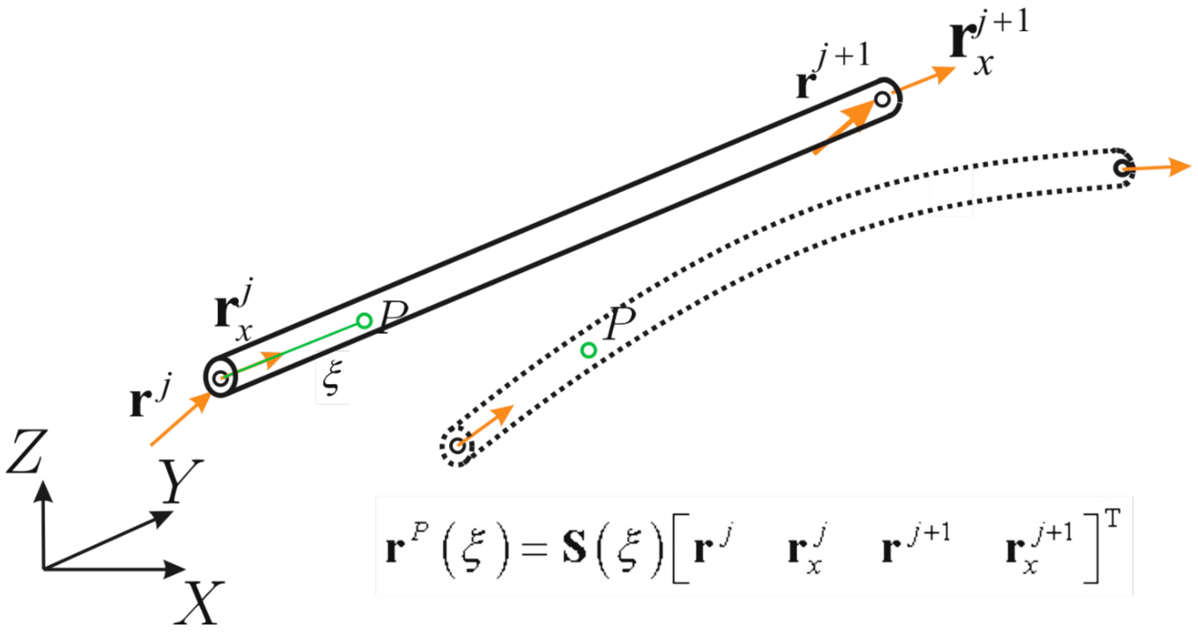
The velocity of any point within the element can be straightforwardly obtained as

$$\underbrace{\dot{\mathbf{r}}(x,y,z,t)}_{\substack{\text{Velocity of an arbitrary}\\ \text{point within the element}}} = \underbrace{\mathbf{S}(x,y,z)}_{\substack{\text{Space-dependent}\\ \text{shape function}}} \underbrace{\dot{\mathbf{q}}(t)}_{\substack{\text{Vector of}\\ \text{generalized velocities}}}$$

The kinetic energy takes, in general, the following expression

$$T = \frac{1}{2}\int_V \rho \dot{\mathbf{r}}^T \dot{\mathbf{r}}\ \mathrm{d}V = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}},\ \text{where } \mathbf{M} = \int_V \rho \mathbf{S}^T \mathbf{S}\ \mathrm{d}V = \text{ constant}$$

ANCF Cable: Virtual work of elastic forces [In Chrono]



$$\mathbf{r}^P(\xi) = \mathbf{S}(\xi)\left[\mathbf{r}^j \quad \mathbf{r}_x^j \quad \mathbf{r}^{j+1} \quad \mathbf{r}_x^{j+1}\right]^{\mathrm{T}}$$

Position field of an ANCF beam element

$$\mathbf{r} = \left[s_1\mathbf{I}_{3\times3} \quad s_2\mathbf{I}_{3\times3} \quad s_3\mathbf{I}_{3\times3} \quad s_4\mathbf{I}_{3\times3}\right]\left[\mathbf{q}_1^{\mathrm{T}} \quad \mathbf{q}_2^{\mathrm{T}}\right] = \mathbf{S}(\xi)\mathbf{q}$$

Beam longitudinal position vector gradient

$$\mathbf{r}_x = \mathbf{S}_x(\xi)\mathbf{q}$$

- Have one position vector gradient pointing along the beam centerline
- Account for axial and bending strains

- Nodal coordinates

$$\mathbf{q}_j(t) = \left[\mathbf{r}_j^{\mathrm{T}} \quad \mathbf{r}_{j,x}^{\mathrm{T}}\right]^T$$

- Shape functions: $\xi = x/l\,,\ \xi = 0,\ldots,1$

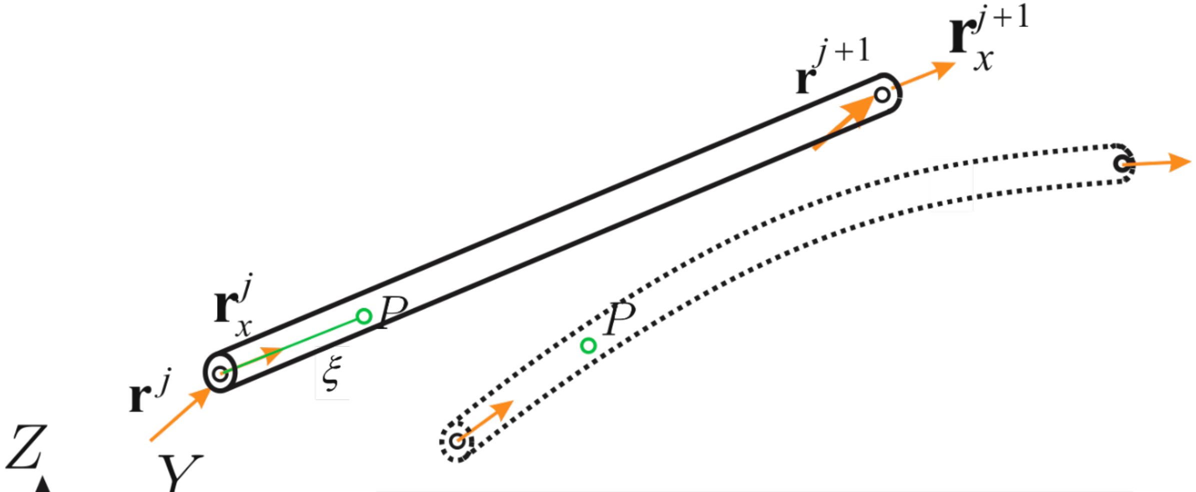
$$s_1 = 1 - 3\xi^2 + 2\xi^3$$

$$s_2 = \boxed{l}\left(\xi - 2\xi^2 + \xi^3\right)$$

$$s_3 = 3\xi^2 - 2\xi^3$$

$$s_4 = \boxed{l}\left(-\xi^2 + \xi^3\right)$$

ANCF Cable: Virtual work of elastic forces



$$\mathbf{r}^{P}(\xi) = \mathbf{S}(\xi)\left[\mathbf{r}^{j} \quad \mathbf{r}_x^{j} \quad \mathbf{r}^{j+1} \quad \mathbf{r}_x^{j+1}\right]^{\mathrm{T}}$$

The virtual of the elastic forces for the gradient-deficient beam element may be defined as

$$\delta W_e = \int_L \left[EA\varepsilon_x \delta\varepsilon_x + EI\kappa\delta\kappa\right]\mathrm{d}x$$

where

$$\varepsilon_x = \frac{1}{2}\left(\mathbf{r}_x^{\mathrm{T}}\mathbf{r}_x - 1\right) \text{ and } \kappa = \frac{\left|\mathbf{r}_x \times \mathbf{r}_{xx}\right|}{\left|\mathbf{r}_x\right|^2}$$

are the axial Green-Lagrange strain and bending curvature

- Calculation of strain variations for virtual work

$$\delta\varepsilon_x = \frac{\partial}{\partial \mathbf{e}}\left(\frac{1}{2}\left(\mathbf{r}_x^{\mathrm{T}}\mathbf{r}_x - 1\right)\right)\delta\mathbf{e} = \mathbf{r}_x^{\mathrm{T}}\frac{\partial \mathbf{r}_x}{\partial \mathbf{e}}\delta\mathbf{e}$$

- For bending curvature:

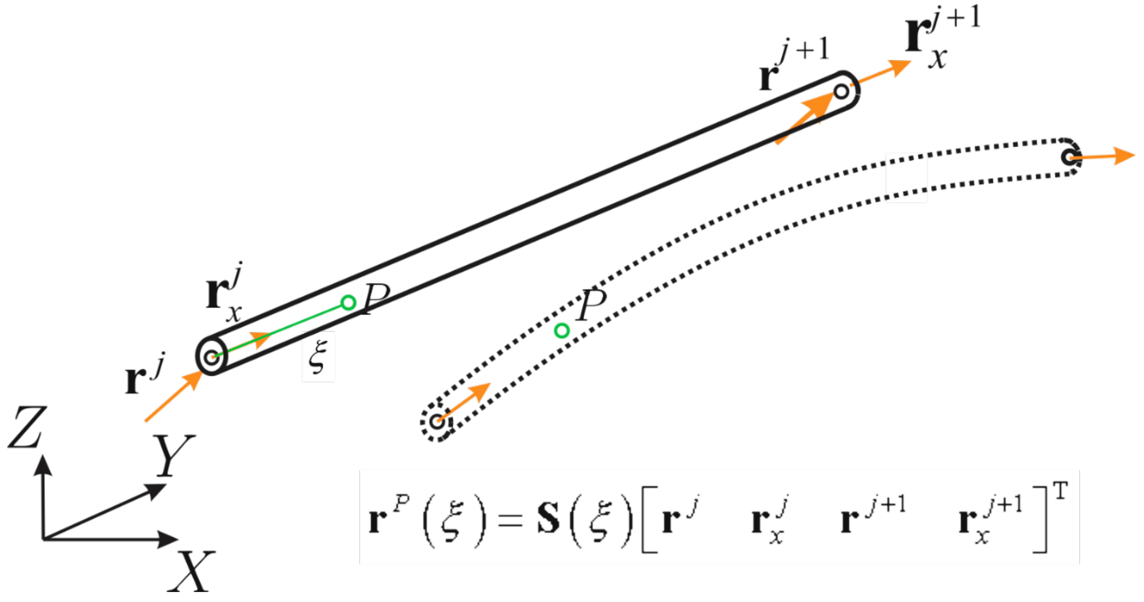
$$\kappa = \frac{f}{g} = \frac{\left|\mathbf{r}_x \times \mathbf{r}_{xx}\right|}{\left|\mathbf{r}_x\right|^3}, \quad f = \left|\mathbf{r}_x \times \mathbf{r}_{xx}\right|, \quad g = \left|\mathbf{r}_x\right|^3$$

$$\delta\kappa = \frac{\partial \kappa}{\partial \mathbf{e}}\delta\mathbf{e} = \frac{1}{g^2}\left(g\frac{\partial f}{\partial \mathbf{e}} - f\frac{\partial g}{\partial \mathbf{e}}\right)\delta\mathbf{e}$$

$$\frac{\partial f}{\partial \mathbf{e}} = \frac{\partial}{\partial \mathbf{e}}\sqrt{\left(\mathbf{r}_x \times \mathbf{r}_{xx}\right)^{\mathrm{T}}\left(\mathbf{r}_x \times \mathbf{r}_{xx}\right)} = \frac{\left(\mathbf{r}_x \times \mathbf{r}_{xx}\right)^{\mathrm{T}}\left(\frac{\partial}{\partial \mathbf{e}}\mathbf{r}_x \times \mathbf{r}_{xx} + \mathbf{r}_x \times \frac{\partial}{\partial \mathbf{e}}\mathbf{r}_{xx}\right)}{\sqrt{\left(\mathbf{r}_x \times \mathbf{r}_{xx}\right)^{\mathrm{T}}\left(\mathbf{r}_x \times \mathbf{r}_{xx}\right)}}$$

$$\frac{\partial g}{\partial \mathbf{e}} = \frac{\partial}{\partial \mathbf{e}}\left(\mathbf{r}_x^{\mathrm{T}}\mathbf{r}_x\right)^{3/2} = 3\left(\mathbf{r}_x^{\mathrm{T}}\mathbf{r}_x\right)^{1/2}\left(\mathbf{r}_x^{\mathrm{T}}\frac{\partial \mathbf{r}_x}{\partial \mathbf{e}}\right)$$

ANCF Cable: Equations of motion



$$\mathbf{r}^P(\xi) = \mathbf{S}(\xi)\begin{bmatrix}\mathbf{r}^j & \mathbf{r}_x^j & \mathbf{r}^{j+1} & \mathbf{r}_x^{j+1}\end{bmatrix}^{\mathrm{T}}$$

The generalized internal force may be written as…

$$\left.\mathbf{Q}_e^i\right|_{12\times 1} = \left.\mathbf{Q}_{e,b}^i\right|_{12\times 1} + \left.\mathbf{Q}_{e,a}^i\right|_{12\times 1} =$$

$$\int_L EI \underbrace{\frac{1}{g^2}\left(g\frac{\partial f}{\partial \mathbf{e}} - f\frac{\partial g}{\partial \mathbf{e}}\right)}_{\frac{\partial \kappa}{\partial \mathbf{e}}} \mathrm{d}\xi + \int_L EA3 \underbrace{\left(\mathbf{r}_x^{\mathrm{T}}\mathbf{r}_x\right)^{1/2}\left(\mathbf{r}_x^{\mathrm{T}}\frac{\partial \mathbf{r}_x}{\partial \mathbf{e}}\right)}_{\frac{\partial \varepsilon_x}{\partial \mathbf{e}}} \mathrm{d}\xi$$

Mass matrix easily obtained from kinetic energy

$$T = \frac{1}{2}\int_V \rho \dot{\mathbf{r}}^T \dot{\mathbf{r}}\ \mathrm{d}V = \frac{1}{2}\dot{\mathbf{e}}^T \mathbf{M}\dot{\mathbf{e}},$$

$$\text{where } \mathbf{M}_{12\times 12} = \rho A \int_V \rho \mathbf{S}^T \mathbf{S}\ \mathrm{d}x = \text{ constant}$$

Gravity forces

$$\delta W_g = \mathbf{F}_g \cdot \delta \mathbf{r} = \mathbf{g}\int_V \rho\ \mathrm{d}V \cdot \mathbf{S}\delta\mathbf{e} = \mathbf{g}m \cdot \mathbf{S}\delta\mathbf{e}$$

$$\mathbf{Q}_g = \mathbf{S}^T(m\mathbf{g})$$

Equations of motion after assembling finite elements

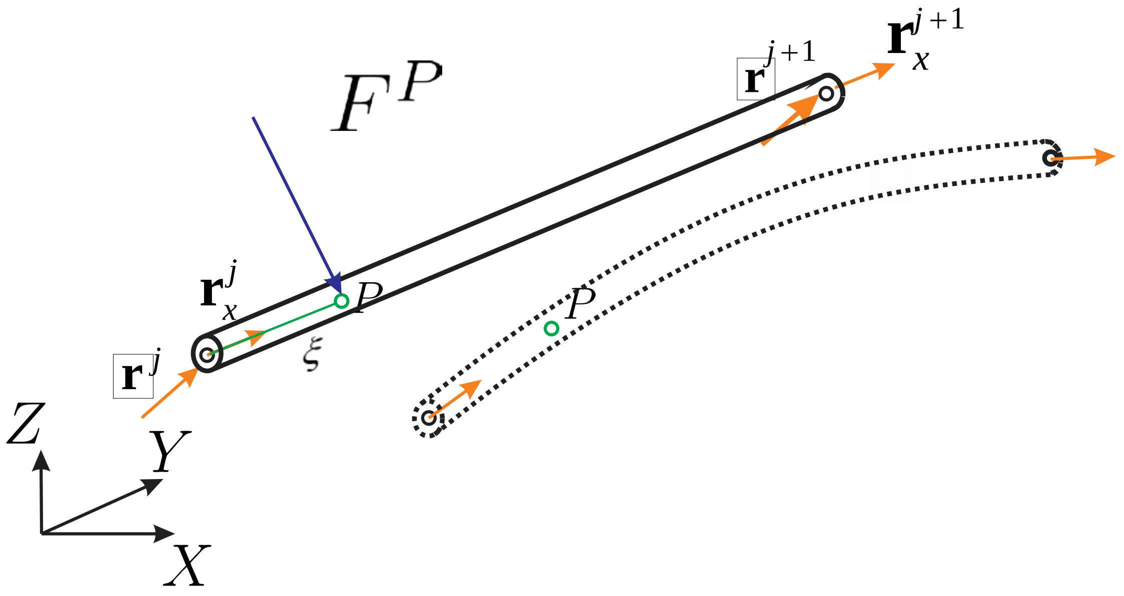
$$\mathbf{M}\ddot{\mathbf{e}} = \mathbf{Q}_e + \mathbf{Q}_g$$

Generalized external forces: ANCF cable

Principle of virtual work may be used to add external forces of diverse nature

- Point forces
- Linear forces, e.g. evenly (or not) distributed pressure
- Volumetric forces

Generalized external forces: Point load



Point load:

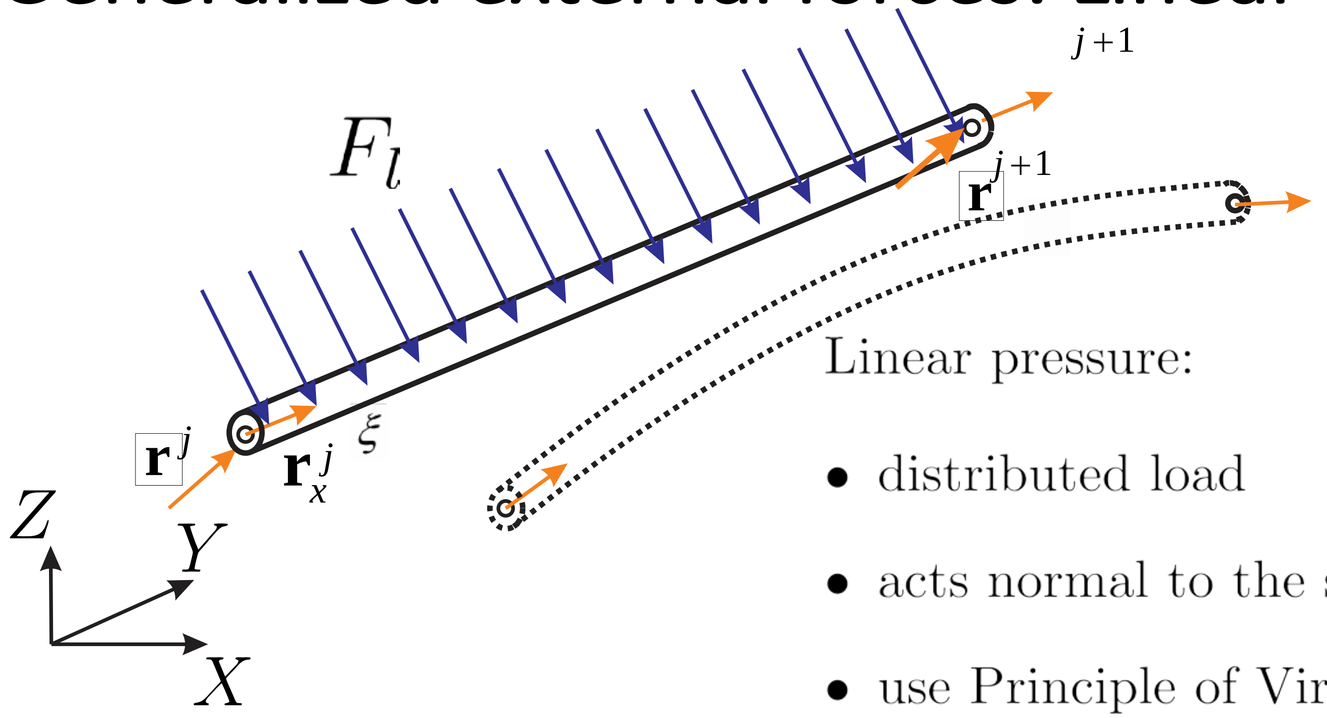
- concentrated load
- acts on one finite element at any point
- does not require numerical integration

$$\delta W_{cl} = \mathbf{F}^{PT}\delta\mathbf{r}^P = \underbrace{\mathbf{Q}_{cl}^T}_{\substack{\text{Generalized}\\\text{force}}} \underbrace{\delta\mathbf{e}}_{\substack{\text{Variation of}\\\text{generalized coordinates}}}$$

$$\mathbf{Q}_{cl} = \mathbf{S}^T\left(\xi_P\right)\boxed{\mathbf{F}^P}$$

Force may depend on time, other coordinates, etc.

Generalized external forces: Linear pressure

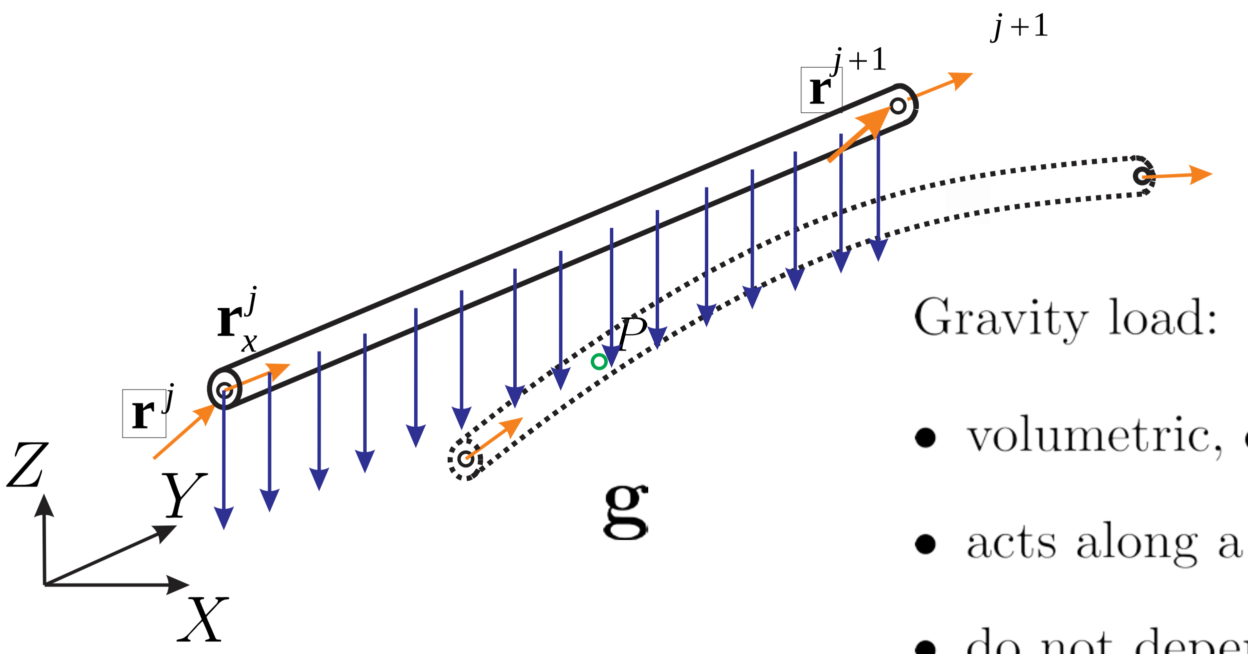


Linear pressure:

- distributed load
- acts normal to the surface
- use Principle of Virtual Work to obtain generalized counterpart

$$\mathbf{Q}_{pres} = -\int_L \underbrace{\mathbf{S}^T(\xi)}_{\text{Shape function}} \overbrace{F_l}^{\substack{\text{constant}\\ \text{linear pressure}}} \underbrace{\boxed{\mathbf{n}}}_{\substack{\text{direction}\\ \text{of load}}} \overbrace{\det[\mathbf{J}]}^{\substack{\text{Det. of Jacobian of}\\ \text{the transformation}}} \mathrm{d}x = \overbrace{\sum_{i=0}^{n_i} w_i \mathbf{S}^T(\xi_i) F_l \mathbf{n}(\xi_i) \det[\mathbf{J}]}^{\text{Numerically solve the integral: Gauss Quadrature}}$$

Generalized external forces: Gravity load



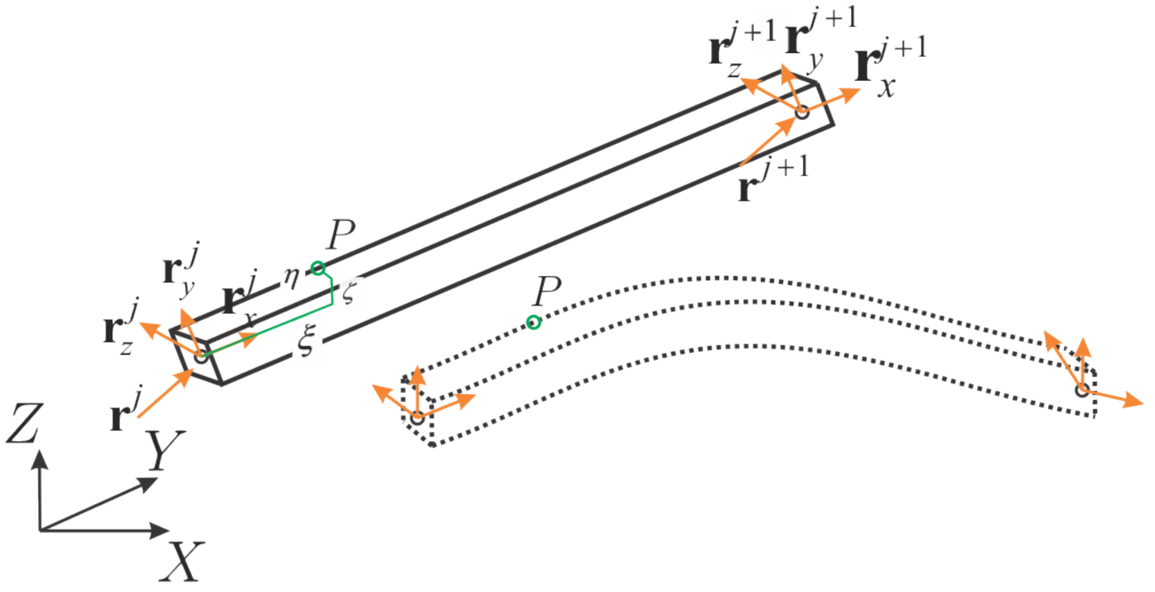
Gravity load:

- volumetric, distributed load
- acts along a global direction
- do not depend on the finite element's coordinates

$$\mathbf{Q}_{grav} = -A \int_L \underbrace{\mathbf{S}^T(\xi)}_{\text{Shape function}} \overbrace{\rho}^{\text{density}} \underbrace{\mathbf{g}}_{\text{acceleration of gravity}} \overbrace{\det[\mathbf{J}]}^{\text{Det. of Jacobian of the transformation}} dx = A \overbrace{\sum_{i=0}^{n_i} w_i \mathbf{S}^T(\xi_i) \rho \mathbf{g} \det[\mathbf{J}]}^{\text{Numerically solve the integral:Gauss Quadrature}}$$

ANCF fully parameterized beam

- Fully parameterized beam: The most straightforward beam element
- 2 nodes; one position vector and three position vector gradients: x, y, and z
- An elastic line approach has been used to alleviate Poisson locking (we'll see more in future lectures)
- It allows for 3D definition of elastic forces
  - Severe locking; i.e. bad convergence properties
    - Correct results not achieved even with fine discretization or
    - Need many elements to obtain correct results

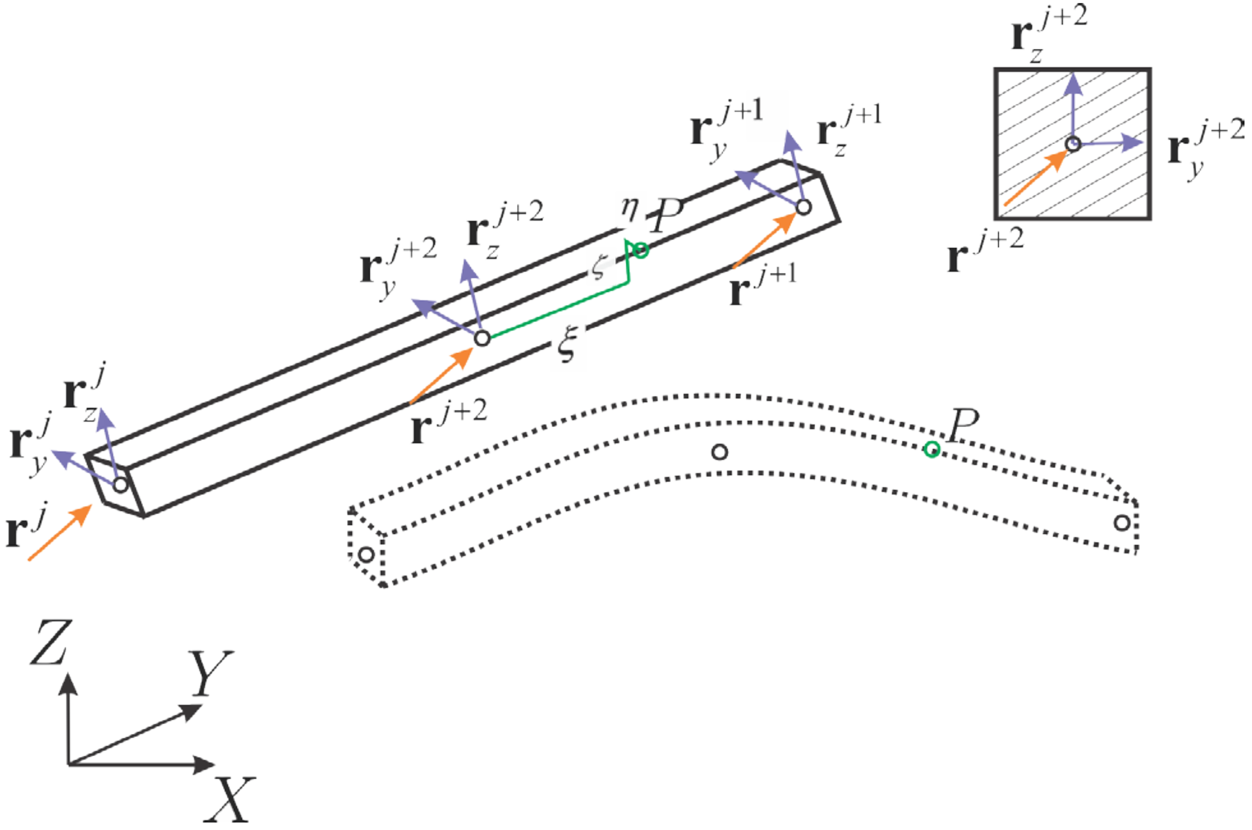


$$\mathbf{r}^P(\xi,\eta,\zeta)=\mathbf{S}(\xi,\eta,\zeta)\begin{bmatrix}\mathbf{r}^j & \mathbf{r}_x^j & \mathbf{r}_y^j & \mathbf{r}_z^j & \mathbf{r}^{j+1} & \mathbf{r}_x^{j+1} & \mathbf{r}_y^{j+1} & \mathbf{r}_z^{j+1}\end{bmatrix}^{\mathrm{T}}$$

Interpolation:

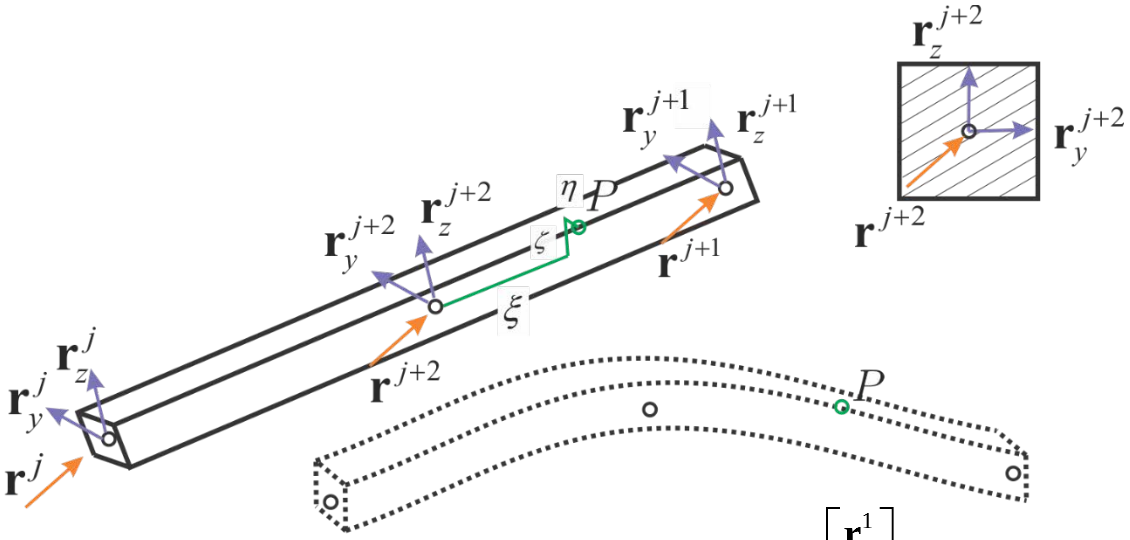
- Linear in y and z (cross section plane)
  - Poisson locking: Poisson effect
- Cubic in longitudinal direction
  - Shear locking
  - FE locking: Excessive (unwanted) stiffness in some FE deformation modes

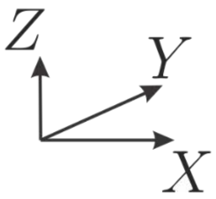
3-node shear deformable ANCF beam [In Chrono]



- Developed in Nachbagauer et al, 2013, “Structural and Continuum Mechanics Approaches for a 3D Shear Deformable ANCF Beam Finite Element: Application to Static and Linearized Dynamic Examples”, JCND, Vol. 8, 021004-1
- Avoid locking issues
- Can use structural (Reissner) and continuum-based approaches
- Describe two bending strains, two shears, torsion, and stretch

3-node shear deformable ANCF beam





$$\mathbf{r} = \begin{bmatrix} s_1\mathbf{I} & s_2\mathbf{I} & \dots & s_9\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}^1 \\ \mathbf{r}_y^1 \\ \dots \\ \mathbf{r}_z^3 \end{bmatrix}$$

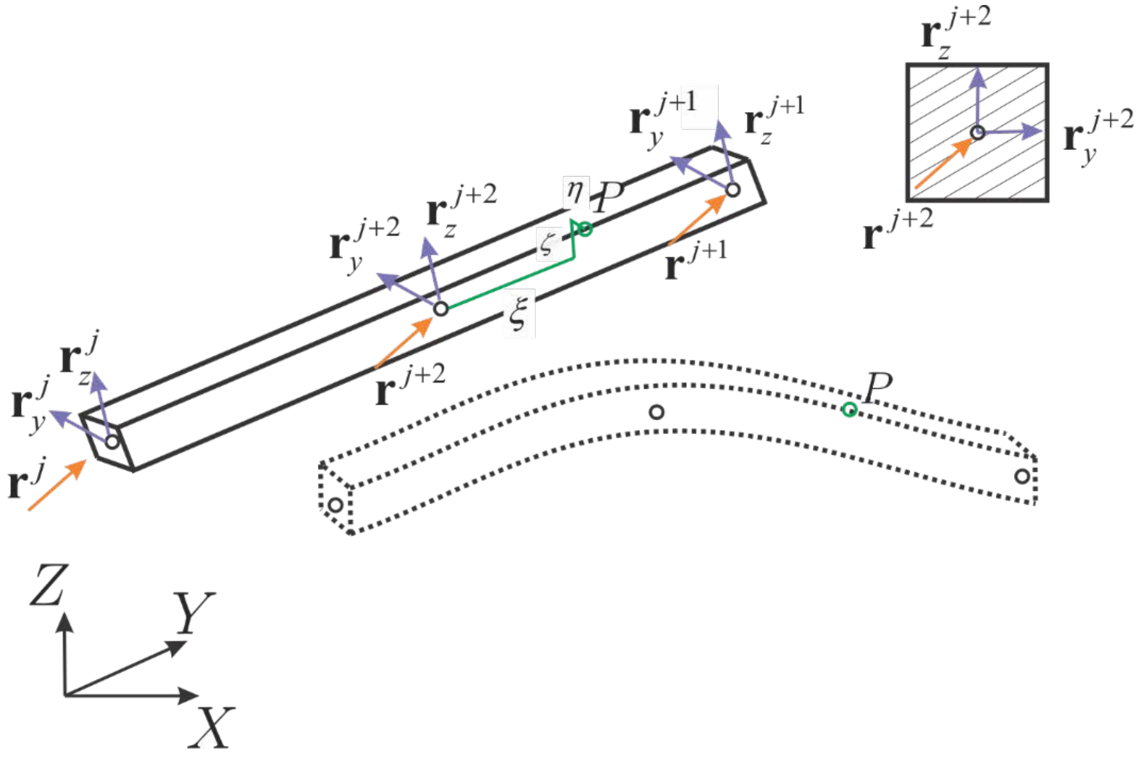
- Quadratic shape functions in longitudinal direction
- Linear interpolation over the cross section
- Reference of SF taken at the center of the element –node 3.

$$s_1 = -\frac{\xi}{2}(1-\xi),\ s_2 = \eta s_1,\ s_3 = \zeta s_1 \text{[First node]}$$

$$s_4 = \frac{\xi}{2}(1+\xi),\ s_5 = \eta s_4,\ s_6 = \zeta s_4 \text{[Second node]}$$

$$s_7 = -(\xi - 1)(\xi + 1),\ s_8 = \eta s_7,\ s_9 = \zeta s_7 \text{[Third node]}$$

3-node shear deformable ANCF beam: Structural mechanics



1) Create a coord. syst. at cross section

$$\mathbf{e}_1 = \frac{\mathbf{e}_1}{|\mathbf{e}_1|},\ \mathbf{e}_1 = \mathbf{r}_y \times \mathbf{r}_z;\ \mathbf{e}_3 = \frac{\mathbf{e}_3}{|\mathbf{e}_3|},\ \mathbf{e}_3 = \mathbf{r}_z;\ \mathbf{e}_2 = \frac{\mathbf{e}_2}{|\mathbf{e}_2|},\ \mathbf{e}_2 = \mathbf{r}_z \times \left(\mathbf{r}_y \times \mathbf{r}_z\right);$$

$$\mathbf{A}_{\mathrm{cs}} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

2) Stretch and shear defined as

$$\Gamma_1 = \mathbf{e}_1^T \mathbf{r}_x - 1,\ \Gamma_2 = \mathbf{e}_2^T \mathbf{r}_x, \Gamma_3 = \mathbf{e}_3^T \mathbf{r}_x,\ \mathbf{r}_x = \partial \mathbf{r} / \partial x$$

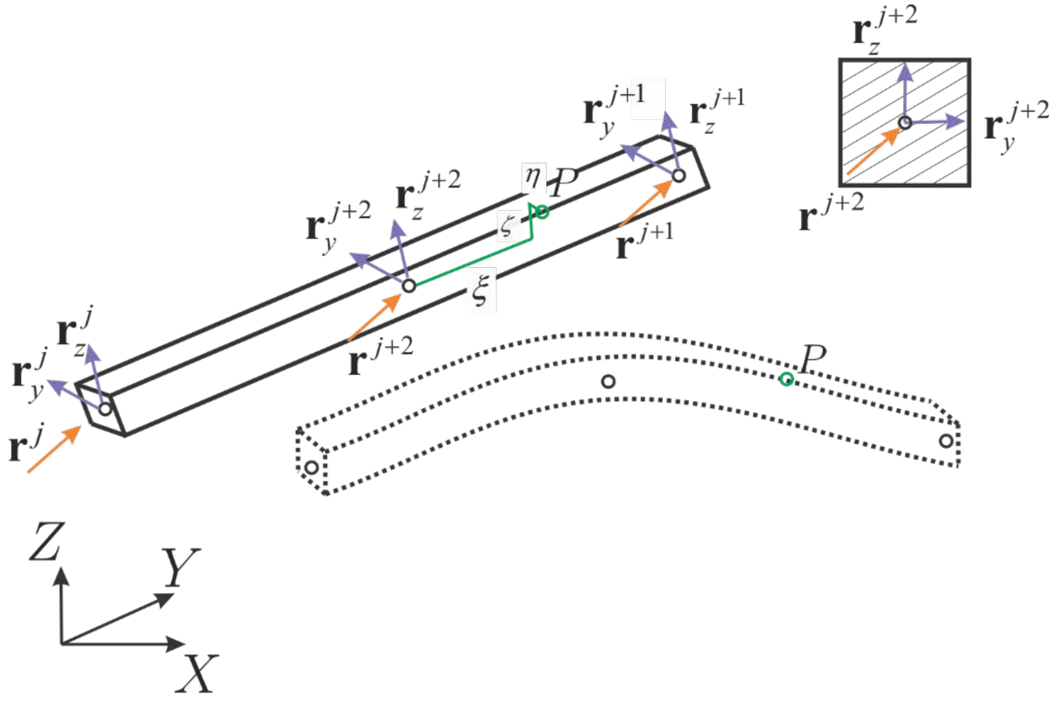
3) Bending and torsion

$$\mathbf{k} = \mathbf{A}^T \mathbf{A}'_0 = \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix},\ \boldsymbol{\kappa} = \mathrm{axial}(\mathbf{k})$$

Using SM approach, the elastic energy of the element is

$$U^{SM} = \frac{1}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{\boxed{\boldsymbol{\Gamma}^T \mathrm{diag}(EA, GAk_2, GAk_3) \boldsymbol{\Gamma}}}_{\text{Stretch and shear}} + \underbrace{\boxed{\boldsymbol{\kappa}^T \mathrm{diag}(GJ, EI_2, EI_3) \boldsymbol{\kappa}}}_{\text{Bending and torsion}} \, \mathrm{d}\xi$$

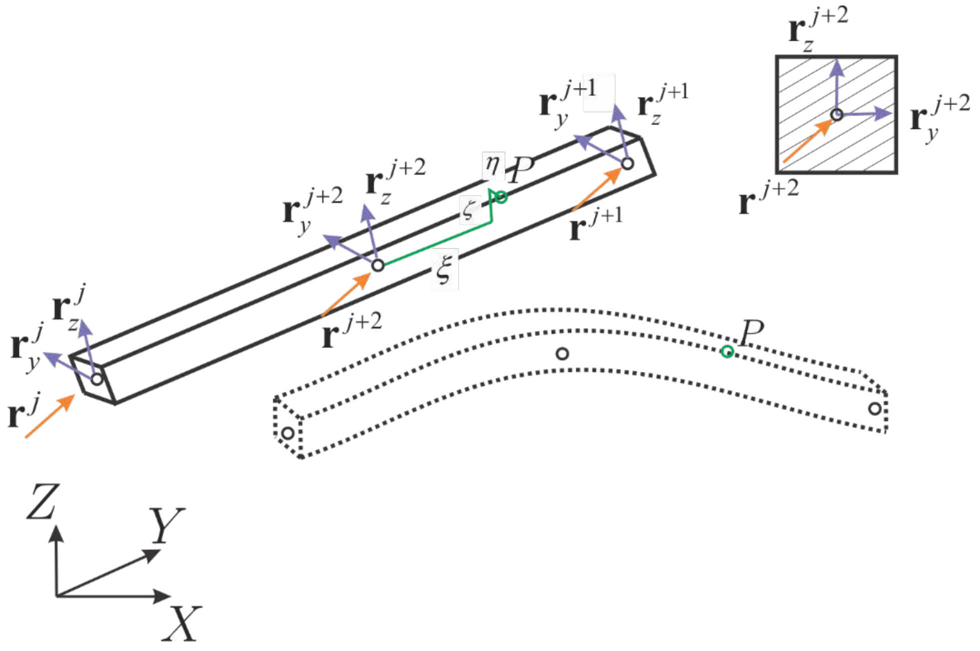
3-node shear deformable ANCF beam: Structural mechanics



- Previous strain measures must be objective (they are!)
- Inertia matrix calculation is straightforward
- Generalized internal force must account for cross section area frame: Not as straightforward as CB approach

- This element has been validated for:
  - Small deformation (analytical solution)
  - Large deformation: Torsional moment (180 deg. twist)
  - Eigenfrequencies: Analytical solution of Timoshenko beam
- It has additional coordinates. Additional cross section elastic energy must be introduced (not accounted for previously)

3-node shear deformable ANCF beam : Continuum-based [In Chrono]



The work of elastic forces can be derived from nonlinear continuum mechanics, using the relation between the nonlinear Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor. The deformation gradient is defined as

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{r}}{\partial \xi}\frac{\partial \xi}{\partial \mathbf{r}_0} = \begin{bmatrix} \frac{\partial r_1}{\partial \xi} & \frac{\partial r_1}{\partial \eta} & \frac{\partial r_1}{\partial \zeta} \\ \frac{\partial r_2}{\partial \xi} & \frac{\partial r_2}{\partial \eta} & \frac{\partial r_2}{\partial \zeta} \\ \frac{\partial r_3}{\partial \xi} & \frac{\partial r_3}{\partial \eta} & \frac{\partial r_3}{\partial \zeta} \end{bmatrix} \boxed{\begin{bmatrix} \frac{\partial r_{01}}{\partial \xi} & \frac{\partial r_{01}}{\partial \eta} & \frac{\partial r_{01}}{\partial \zeta} \\ \frac{\partial r_{02}}{\partial \xi} & \frac{\partial r_{02}}{\partial \eta} & \frac{\partial r_{02}}{\partial \zeta} \\ \frac{\partial r_{03}}{\partial \xi} & \frac{\partial r_{03}}{\partial \eta} & \frac{\partial r_{03}}{\partial \zeta} \end{bmatrix}^{-1}}$$

Accounts for distorted initial configurations

3-node shear deformable ANCF beam: Continuum-based

Possibly distorted reference, captured by $\mathbf{J} = \frac{\partial \mathbf{r}_0}{\partial \xi}$
The strain strain relation is given by $\sigma = \mathbf{D}\varepsilon$ where $\mathbf{D}$ is the elastic matrix and ε the Green-Lagrange strain vector.

The elasticity matrix is given (for isotropic materials), as:

$$\mathbf{D} = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{1-\nu}{\nu} & 1 & 1 & 0 & 0 & 0 \\ 1 & \frac{1-\nu}{\nu} & 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{1-\nu}{\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\frac{1-2\nu}{2\nu}k_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{\frac{1-2\nu}{2\nu}k_3} \end{bmatrix}$$

Become shear modulus G

where theory of thick beams (Timoshenko) has been assumed –k_2 and k_3 are Timoshenko shear correction factors, dependent on beam cross section shape and material properties.

3-node shear deformable ANCF beam: Continuum-based

The elastic energy, to be used to obtain generalized internal forces, may then be written as:

$$U^{CB} = \frac{1}{2} \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma} \det(\mathbf{J})\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

Poisson ratio ν couples ε_{xx} with ε_{yy} and ε_{zz}. This coupling if integrated over the volume of this element causes unwanted stiffness of the bending mode -that is, locking. To avoid this, one type of selective integration is used:

$$U^{CB}_{SI} = \frac{1}{2} \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{D}^{0} \boldsymbol{\varepsilon} \det(\mathbf{J})\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta + \frac{1}{2} HW \int_{-L/2}^{L/2} \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{D}^{\nu} \boldsymbol{\varepsilon} \det(\mathbf{J})\, \mathrm{d}\xi$$

In which

$$\mathbf{D} = \underbrace{\mathbf{D}^{0}}_{\substack{\text{Matrix of elastic coeff.} \\ \text{with no Poisson effect}}} + \underbrace{\mathbf{D}^{\nu}(\nu)}_{\substack{\text{Matrix of elastic coeff.} \\ \text{with Poisson effect}}}$$

- Elastic forces are integrated using Gauss quadrature, as usual
- This element, in the continuum-based flavor is available in Chrono
- It captures many more modes than the “cable” element

3. ANCF shells

Shells

"In many problems of deformation of shells the bending stresses can be neglected, and only the stresses due to strain in the middle surface of the shell need be considered. Take, as an example, a thin spherical container submitted to the action of a **uniformly distributed internal pressure normal to the surface of the shell**. Under this action, the middle surface of the shell undergoes a uniform strain; and since the thickness of the shell is small, the tensile stresses can be assumed as uniformly distributed across the thickness."

Examples

- Balloon
- Pressurized tank
- Tire (in many scenarios)

ANCF shell elements

In terms of formulations of finite elements:

- Plate elements look for capturing **bending curvature**
- Shell elements are based on **in-plane strains**

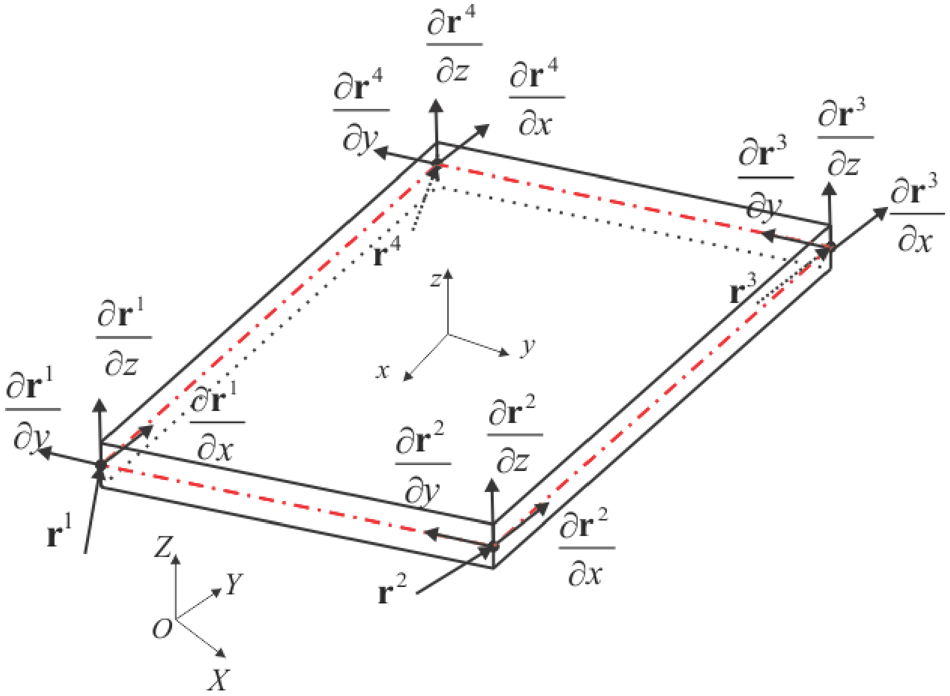
ANCF shell/plate elements may be categorized according to

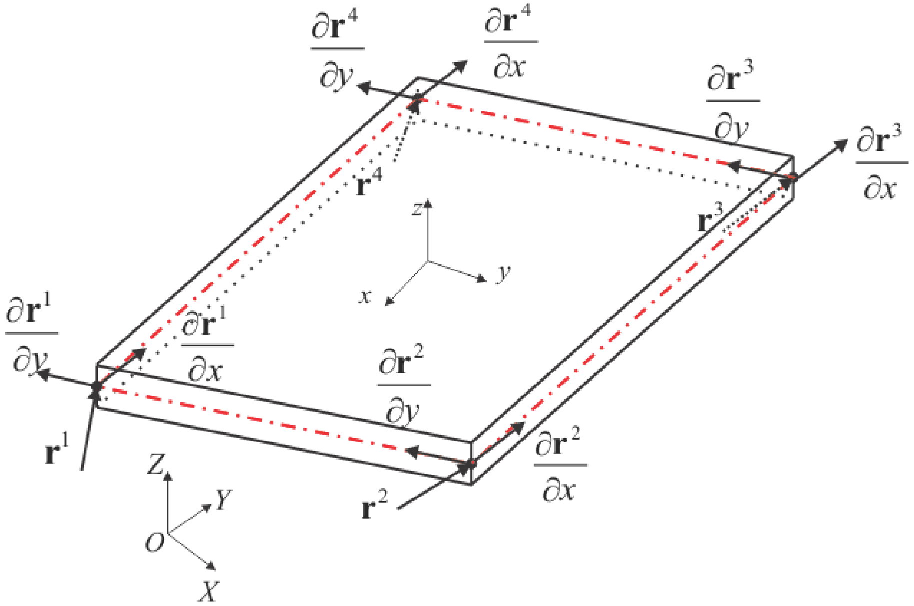
- Whether tranverse shear is considered (**thick** or **thin**)
- The set of position gradient vectors used

ANCF shell/plate elements may also suffer from a variety of locking phenomena

- Shear
- Volumetric/Poisson

ANCF shell elements. Types





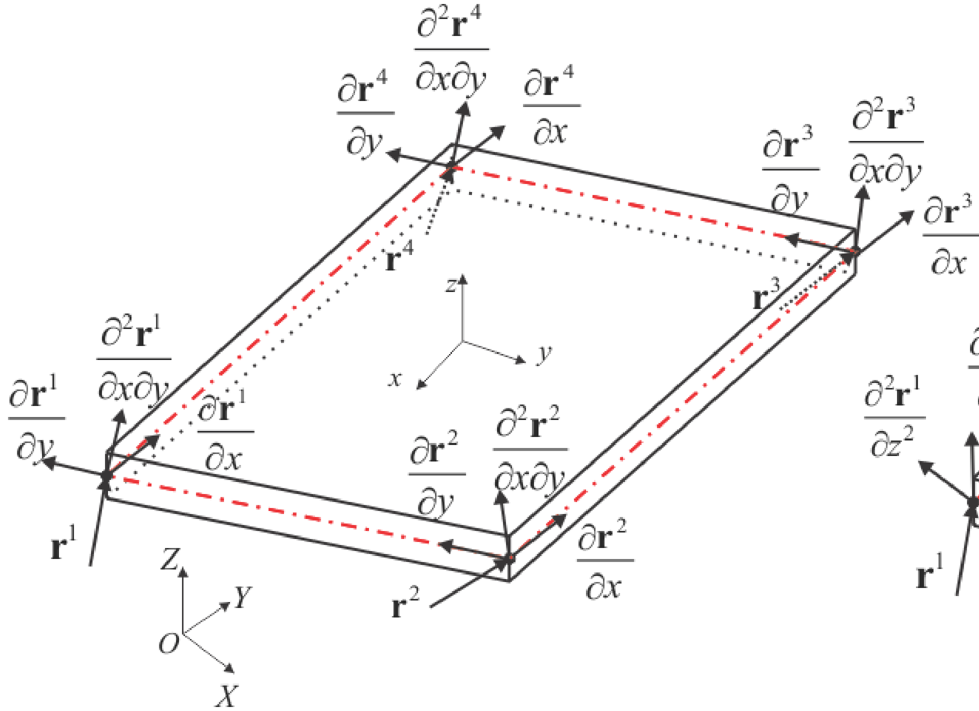
- Fully parameterized
- Gradients are continuous
- Continuum-based
- Locking

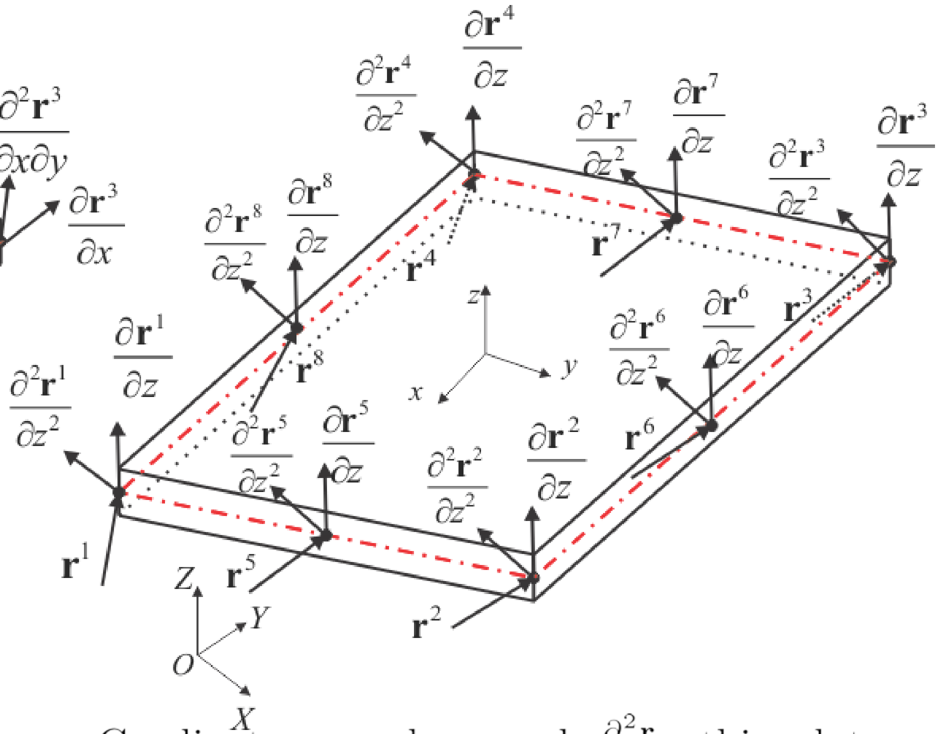
- Gradients $\mathbf{r}_x$ and $\mathbf{r}_y$, thin plate
- Gradients are discontinuous
- Structural based – plate formulation

$$\kappa_{\text{thin}} = \left[\mathbf{n}^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_{\text{mid}}}{\partial x \partial x} \quad \mathbf{n}^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_{\text{mid}}}{\partial y \partial y} \quad \mathbf{n}^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_{\text{mid}}}{\partial x \partial y} \right]^{\mathrm{T}}$$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}$$

ANCF shell elements. Types





- Gradients $\mathbf{r}_x$ and $\mathbf{r}_y$, and $\frac{\partial^2 \mathbf{r}}{\partial x \partial y}$, thin plate
- Gradients are continuous
- Structural based – plate formulation

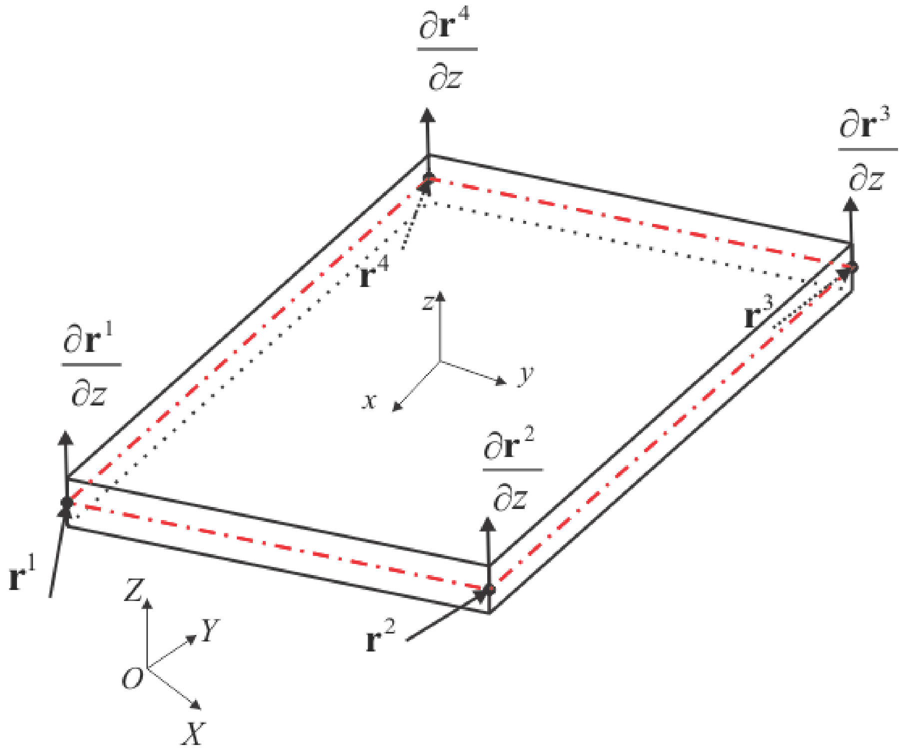
$$\kappa_{\text{thin}} = \left[\mathbf{n}^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_{\text{mid}}}{\partial x \partial x} \quad \mathbf{n}^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_{\text{mid}}}{\partial y \partial y} \quad \mathbf{n}^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_{\text{mid}}}{\partial x \partial y} \right]^{\mathrm{T}}$$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}$$

- Gradients $\mathbf{r}_x$ and $\mathbf{r}_y$, and $\frac{\partial^2 \mathbf{r}}{\partial x \partial y}$, thin plate
- 8 nodes
- Continuum-based: St. Venant-Kirchhoff material
- Extends membrane theory to out of plane strains

$$\mathbf{E} = 0.5 \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})$$

Chrono's ANCF shell element: Bilinear



- Gradient $\mathbf{r}_z$, thick plate – integration over volume
- 4 Nodes
- Bilinear: Product of linear shape functions
  $s_1 = \frac{1}{4}(1-\xi)(1-\eta)$ $s_2 = \frac{1}{4}(1+\xi)(1-\eta)$
  $s_3 = \frac{1}{4}(1+\xi)(1+\eta)$ $s_4 = \frac{1}{4}(1-\xi)(1+\eta)$
- Suffers from locking if not alleviated
- Continuum-based approach
- Includes out-of-plane strains

Kinematics of the element:

$$\mathbf{r}(\xi,\eta,t) = \underbrace{\mathbf{r}_m(\xi,\eta,t)}_{\text{Position of mid-plane}} \underbrace{+ z \overbrace{\frac{\partial \mathbf{r}}{\partial z}(\xi,\eta,t)}^{\text{Fiber direction}}}_{\text{Position on shell thickness}}$$

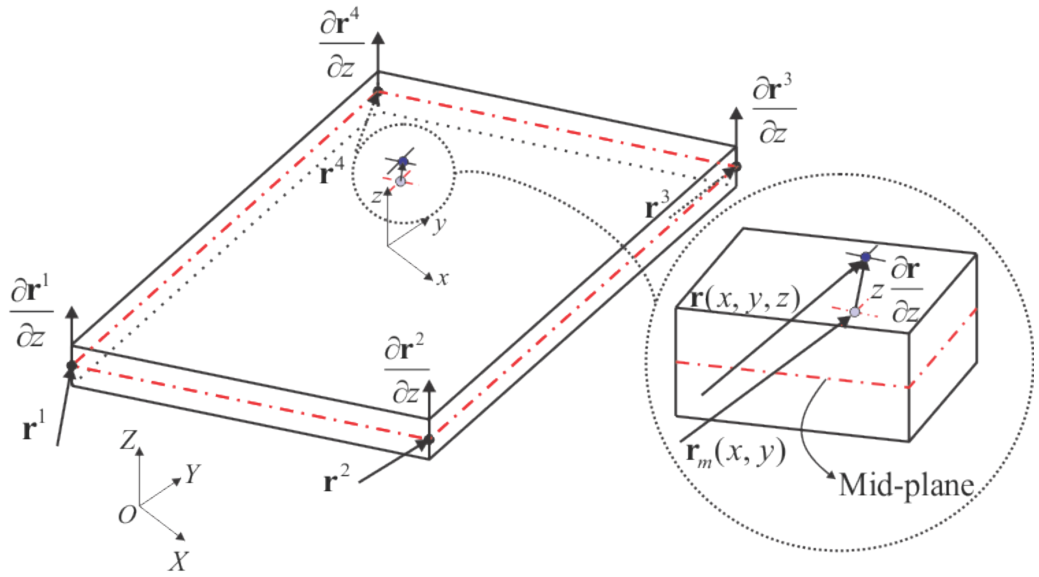
http://sbel.wisc.edu/documents/TR-2016-11.pdf

Chrono’s ANCF shell element

Note that shape functions, position vector gradients, angles, transformation matrices, intermediate operations between frames of reference, and strains are adimensional. The position of an arbitrary point in the shell may be described as

$$\mathbf{r}^i(x^i, y^i, z^i) = \mathbf{S}^i(x^i, y^i, z^i)\mathbf{e}^i,$$

where the combined shape function matrix is given by $\mathbf{S}^i = [\mathbf{S}^i_m \quad z^i\mathbf{S}^i_m]$. Similarly, the coordinates of the element may be grouped together as $\mathbf{e}^i = [(\mathbf{e}^i_p)^T \quad (\mathbf{e}^i_g)^T]^T$, where $(\mathbf{e}^i_p)$ $(\mathbf{e}^i_g)$ are the element position and gradient coordinates.



Chrono's ANCF shell element

Relying on this kinematic description of the shell element, the Green-Lagrange strain tensor may be calculated as

Element *i*

$$\mathbf{E}^{\boxed{i}} = \frac{1}{2}\left(\left(\mathbf{F}^i\right)^T \mathbf{F}^i - \mathbf{I}\right),$$

where $\mathbf{F}^i$ is the deformation gradient matrix defined as the current configuration over the reference configuration. Using the current absolute nodal coordinates, this matrix may be defined as (chain rule)

$$\mathbf{F}^i = \frac{\partial \mathbf{r}^i}{\partial \mathbf{X}^i} = \frac{\partial \mathbf{r}^i}{\partial \mathbf{x}^i}\left(\frac{\partial \mathbf{X}^i}{\partial \mathbf{x}^i}\right)^{-1}$$

The strain tensor can then expressed in vector form in the following manner

$$\boldsymbol{\varepsilon}^i = \left[\boxed{\varepsilon^i_{xx} \quad \varepsilon^i_{yy} \quad \gamma^i_{xy}} \quad \varepsilon^i_{zz} \quad \gamma^i_{xz} \quad \gamma^i_{yz}\right]^T$$

Membrane strains

Chrono’s ANCF shell element

The elastic internal forces are spatially integrated over the element volume using Gaussian quadrature:

$$\mathbf{Q}_k^i = -\int_{V_0} \left(\frac{\partial \varepsilon^c}{\partial \mathbf{e}^i}\right) \frac{\partial W^i(\varepsilon^c + \varepsilon^{EAS})}{\partial \varepsilon^i} \mathrm{d}V_0$$

where ε^c is the compatible strain, obtained from the displacement field using “Assumed Natural Strain” interpolation to avoid transver/in-plane shear. Further, the term $\boxed{W^i(\varepsilon^c + \varepsilon^{EAS})}$ denotes the strain energy density function, which must be obtained by adding an enhanced strain contribution, ε^{EAS}. The second Piola–Kirchhoff stress tensor is obtained from the relation $\boxed{\sigma^i = \frac{\partial W^i(\varepsilon^c + \varepsilon^{EAS})}{\partial \varepsilon^i}}$. The addition of assumed natural strains and enhanced strains finds justifications of the mixed variational principle by Hu–Washizu

Shell strains: Orthotropic and curvilinear reference

Due to manufacturing processes, the initial configuration isn't always "straight"; initial curved geometries of the flexible bodies need to be considered

$$\mathbf{F}^i = \frac{\partial \mathbf{r}^i}{\partial \mathbf{X}^i} = \frac{\partial \mathbf{r}^i}{\partial \mathbf{x}^i}\left(\frac{\partial \mathbf{X}^i}{\partial \mathbf{x}^i}\right)^{-1}$$

$\mathbf{r}^i$: Current configuration; $\mathbf{X}^i$: Initial configuration; $\mathbf{x}^i$: Element reference

The tensor $\mathbf{J}^i = \left(\frac{\partial \mathbf{X}^i}{\partial \mathbf{x}^i}\right) = \frac{\partial\left(\mathbf{S}\mathbf{e}_0^i\right)}{\partial \mathbf{x}^i}$ is constant and can be inverted. The gradient tensor $\mathbf{F}^i$ defines strains in the global frame from a straight configuration as $\mathbf{E}^i = 0.5 \cdot (\mathbf{F}^{i\mathrm{T}}\mathbf{F}^i - \mathbf{I})$

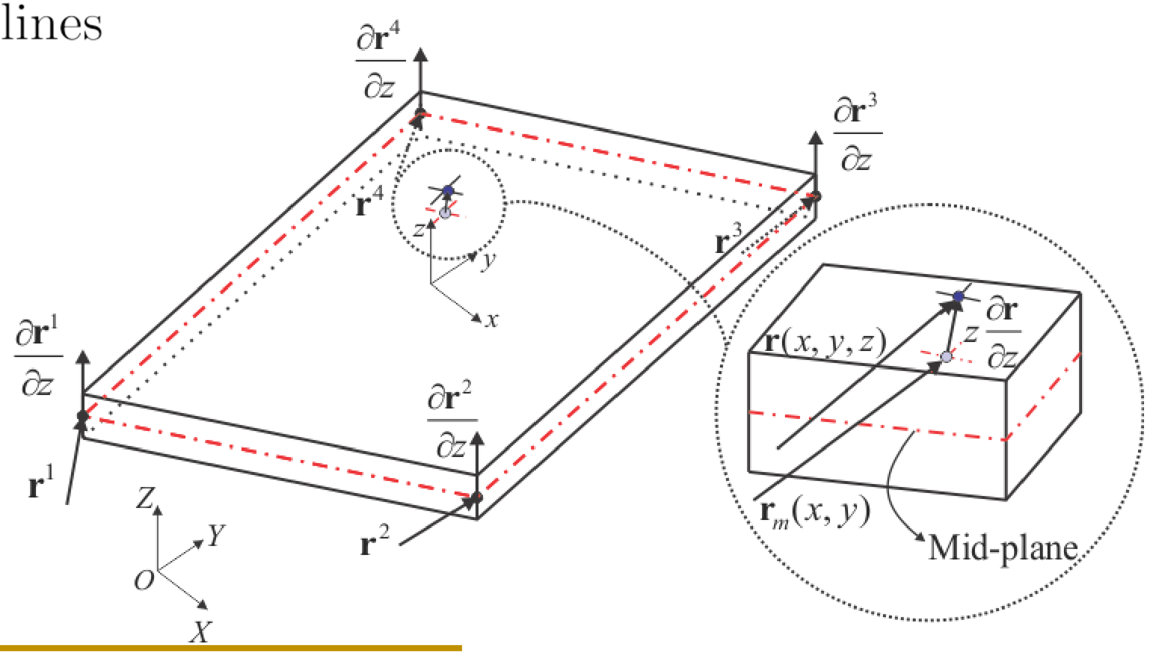
Shell strains: Orthotropic and curvilinear reference

In the curved initial configuration, the position of a material point in the shell is given by

$$\mathbf{r}_0(x,y,z) = \boxed{\mathbf{r}_{m0}}(x,y) + \boxed{z\mathbf{r}_{z0}}(x,y)$$

The element local coordinate system is a Lagrangian coordinate system in which strains are to be measured. This frame of reference defines covariant base vectors along the three curvilinear coordinate lines

$$\begin{aligned}
(\mathbf{g}_0)_1 &= \frac{\partial \mathbf{r}_0}{\partial x} = \frac{\partial \mathbf{r}_{m0}}{\partial x}(x,y) + z\frac{\partial \mathbf{r}_{0z}}{\partial x}(x,y) \\
(\mathbf{g}_0)_2 &= \frac{\partial \mathbf{r}_0}{\partial y} = \frac{\partial \mathbf{r}_{m0}}{\partial y}(x,y) + z\frac{\partial \mathbf{r}_{0z}}{\partial y}(x,y) \\
(\mathbf{g}_0)_3 &= \frac{\partial \mathbf{r}_0}{\partial z} = \mathbf{r}_{0z}(x,y)
\end{aligned}$$



Shell strains: Orthotropic and curvilinear reference

The covariant base vector along the coordinate lines in the current configuration is given by

$$
\begin{aligned}
\boxed{(\mathbf{g})_1} &= \frac{\partial \mathbf{r}}{\partial x} = \frac{\partial \mathbf{r}_m}{\partial x}(x,y) + z\frac{\partial \mathbf{r}_z}{\partial x}(x,y) \\
\boxed{(\mathbf{g})_2} &= \frac{\partial \mathbf{r}}{\partial y} = \frac{\partial \mathbf{r}_m}{\partial y}(x,y) + z\frac{\partial \mathbf{r}_z}{\partial y}(x,y) \\
\boxed{(\mathbf{g})_3} &= \frac{\partial \mathbf{r}}{\partial z} = \mathbf{r}_z(x,y)
\end{aligned}
$$

Current deformed reference

Each component of the covariant Green strain tensor in the **curvilinear** system is defined as

$$E_{IJ} = \frac{1}{2}\left(C_{IJ} - C^0_{IJ}\right),$$

where $C_{IJ} = (\mathbf{g})_I \cdot (\mathbf{g})_J$ and $C^0_{IJ} = (\mathbf{g}_0)_I \cdot (\mathbf{g}_0)_J$

Shell strains: Orthotropic and curvilinear reference

To define the Green strain tensor regularized to its orthonormal frame, we first orthonormalize the covariant base at the initial configuration

$$(\mathbf{e}_0)_1 = \frac{(\mathbf{g}_0)_1}{|(\mathbf{g}_0)_1|}$$
$$(\mathbf{e}_0)_3 = |\mathbf{r}_z|$$
$$(\mathbf{e}_0)_2 = (\mathbf{e}_0)_3 \times (\mathbf{e}_0)_1$$

This frame defines the actual frame in which the material properties are defined. If the material is orthotropic, directions in the material can be considered. For example,

$$(\mathbf{e}_0)_1^{Or} = (\mathbf{e}_0)_1 \cos\theta + (\mathbf{e}_0)_2 \sin\theta$$
$$(\mathbf{e}_0)_3^{Or} = (\mathbf{e}_0)_3$$
$$(\mathbf{e}_0)_2^{Or} = -(\mathbf{e}_0)_1 \sin\theta + (\mathbf{e}_0)_2 \cos\theta$$

In which, theta is the angle θ defines a principal direction in the material (used in tires: steel belts).

Shell strains: Orthotropic and curvilinear reference

In matrix form, the coefficients of contravariance transformation may be obtained from the Jacobian of the position vectors at the reference configuration and the local Cartesian frame including anisotropy in the following form

$$\boldsymbol{\beta} = \begin{bmatrix} \mathbf{Y}^{-1}\big|_{C1}^{\mathrm{T}} \\ \mathbf{Y}^{-1}\big|_{C2}^{\mathrm{T}} \\ \mathbf{Y}^{-1}\big|_{C3}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} (\mathbf{e}_0)_1^{\mathrm{Or}} & (\mathbf{e}_0)_2^{\mathrm{Or}} & (\mathbf{e}_0)_3^{\mathrm{Or}} \end{bmatrix}$$

where $\mathbf{Y}^{-1}|_{Ci}$ is the i column of the inverse of $\mathbf{Y} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \begin{bmatrix} (\mathbf{g}_0)_1 & (\mathbf{g}_0)_2 & \mathbf{n}_0 \end{bmatrix}$. The components of the 3-by-3 matrix $\boldsymbol{\beta}$ are used to set up a transformation matrix necessary for the calculation of strains:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}$$

where $\beta_{ij} = \beta(i, j)$.

Shell strains: Orthotropic and curvilinear reference

Finally the compatible strains are calculated as:

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}} \left(\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} - \begin{bmatrix} (g_0)_{11} & (g_0)_{12} & (g_0)_{13} \\ (g_0)_{21} & (g_0)_{22} & (g_0)_{23} \\ (g_0)_{31} & (g_0)_{32} & (g_0)_{33} \end{bmatrix} \right) \boldsymbol{\beta} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix},}$$

where $g_{ij} = (\mathbf{g})_i \cdot (\mathbf{g})_j$ and $(g_0)_{ij} = (\mathbf{g}_0)_i \cdot (\mathbf{g}_0)_j$.

Final expression! These strains are related to constitutive equations

Shell strains: Orthotropic and curvilinear reference

A few words about the derivations in previous slides

- It is general: No membrane-strain assumption, i.e. fully 3D
- It must be accounted for in the numerical implementation
- It must be considered when obtaining the **generalized** elastic/material forces

Generalized forces –continuum based approach

The energy for a linear elastic material:

$$U_e = \int_V \boldsymbol{\varepsilon}^{\mathrm{T}} \overbrace{\boldsymbol{E}\boldsymbol{\varepsilon}}^{\sigma} \, \mathrm{d}V_0 \Rightarrow \mathbf{Q}_e = \frac{\partial U_e}{\partial \mathbf{e}}$$

Generalized internal forces...

$$\mathbf{Q}_e^i = \int_{V^i} \frac{\partial \boldsymbol{\varepsilon}^{i\mathrm{T}}}{\partial \mathbf{e}} \mathbf{E}^i \boldsymbol{\varepsilon}^i \mathrm{d}V_0^i = \int_{V^i} \frac{\partial \boldsymbol{\varepsilon}^{i\mathrm{T}}\left(\mathbf{e}_0^i, \mathbf{e}^i, \boldsymbol{\beta}^i\right)}{\partial \mathbf{e}} \mathbf{E}^i \boldsymbol{\varepsilon}^i \left(\mathbf{e}_0^i, \mathbf{e}^i, \boldsymbol{\beta}^i\right) \mathrm{d}V_0^i =$$

$$\int_{V^i} \frac{\partial \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} & \varepsilon_{zz} & \varepsilon_{xz} & \varepsilon_{yz} \end{bmatrix}}{\partial \mathbf{e}} \mathbf{E}^i \boldsymbol{\varepsilon}^i \left(\mathbf{e}_0^i, \mathbf{e}^i, \boldsymbol{\beta}^i\right) \mathrm{d}V_0^i$$

where $\boldsymbol{\sigma}$ is the second Piola-Kirchhoff stress tensor, and $\mathrm{d}V_0^i$ is the (infinitesimal) element volume at the initial configuration

- Integrals are integrated numerically using Gauss points
- Order of integration depends on order of integrand (in turn, depends on shape functions')

Full and reduced

Generalized forces –continuum based approach

$$\frac{\partial \varepsilon}{\partial \mathbf{e}} = \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}} \frac{\partial}{\partial \mathbf{e}} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \boldsymbol{\beta} =$$

$$\frac{\partial \varepsilon}{\partial \mathbf{e}} = \frac{1}{2}\boxed{\boldsymbol{\beta}^{\mathrm{T}}} \frac{\partial}{\partial \mathbf{e}} \begin{bmatrix} \mathbf{e}^{\mathrm{T}}\mathbf{S}_x^{\mathrm{T}}\mathbf{S}_x\mathbf{e} & \mathbf{e}^{\mathrm{T}}\mathbf{S}_y^{\mathrm{T}}\mathbf{S}_y\mathbf{e} & \mathbf{e}^{\mathrm{T}}\mathbf{S}_x^{\mathrm{T}}\mathbf{S}_y\mathbf{e} & \mathbf{e}^{\mathrm{T}}\mathbf{S}_z^{\mathrm{T}}\mathbf{S}_z\mathbf{e} & \mathbf{e}^{\mathrm{T}}\mathbf{S}_x^{\mathrm{T}}\mathbf{S}_z\mathbf{e} & \mathbf{e}^{\mathrm{T}}\mathbf{S}_y^{\mathrm{T}}\mathbf{S}_z\mathbf{e} \end{bmatrix} \boldsymbol{\beta} =$$

$$\varepsilon_d = \frac{\partial \varepsilon}{\partial \mathbf{e}} = \frac{1}{2}\boxed{\boldsymbol{\beta}'^{\mathrm{T}}} \begin{bmatrix} \mathbf{S}_x^{\mathrm{T}}\mathbf{S}_x\mathbf{e} & \mathbf{S}_y^{\mathrm{T}}\mathbf{S}_y\mathbf{e} & \mathbf{S}_x^{\mathrm{T}}\mathbf{S}_y\mathbf{e} & \mathbf{S}_z^{\mathrm{T}}\mathbf{S}_z\mathbf{e} & \mathbf{S}_x^{\mathrm{T}}\mathbf{S}_z\mathbf{e} & \mathbf{S}_y^{\mathrm{T}}\mathbf{S}_z\mathbf{e} \end{bmatrix} \boldsymbol{\beta}'$$

Beta matrices need manipulations to accommodate strain vector

Jacobian of internal forces

Jacobian of internal forces is needed when implicit numerical integration is needed

- Computationally more demanding than internal forces
- Accurate enough approximation needed for convergence

$$\boxed{\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{\varepsilon}} \Rightarrow \frac{\partial \mathbf{Q}_e}{\partial \mathbf{e}} = \frac{\partial^2}{\partial \mathbf{e}^2}\left(\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{\varepsilon}\right) = \left(\underbrace{\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\frac{\partial^2 \boldsymbol{\varepsilon}^{\mathrm{T}}}{\partial \mathbf{e}^2}}_{\substack{\text{Algebraic manupulations}\\ \text{needed for this term}}} + \frac{\partial \boldsymbol{\varepsilon}^{\mathrm{T}}}{\partial \mathbf{e}}\mathbf{E}\frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{e}} \right)$$

Integrand of elastic energy

- More details on this, in Chrono implementation
- $\mathbf{E}$ is a matrix of elastic coefficients that contain moduli of elasticity, rigidity, and Poisson ratios in the 3 directions: E_x, E_y, E_z, G_x, G_y, G_z, ν_x, ν_y, ν_z

Mass matrix

The mass matrix of the element is given by

$$\mathbf{M}^i = \int_{V_o^i} \rho_0^i (\mathbf{S}^i)^T \mathbf{S}^i \mathrm{d}V_o^i,$$

which remains constant throughout the simulation. The equations of motion may be written as

$$\mathbf{M}^i \ddot{\mathbf{e}}^i = \mathbf{Q}_k^i(\mathbf{e}^i, \dot{\mathbf{e}}^i, \boxed{\boldsymbol{\alpha}^i)} + \mathbf{Q}_e^i(\mathbf{e}^i, \dot{\mathbf{e}}^i, t),$$

Internal parameters

where $\mathbf{Q}_k$ is the element elastic force vector and $\mathbf{Q}_e$ is the external force vector.

Generalized external forces

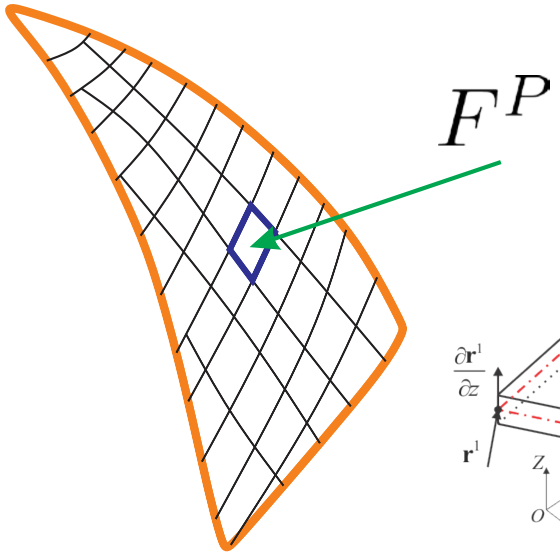
Principal of virtual work may be used to add external forces of diverse nature

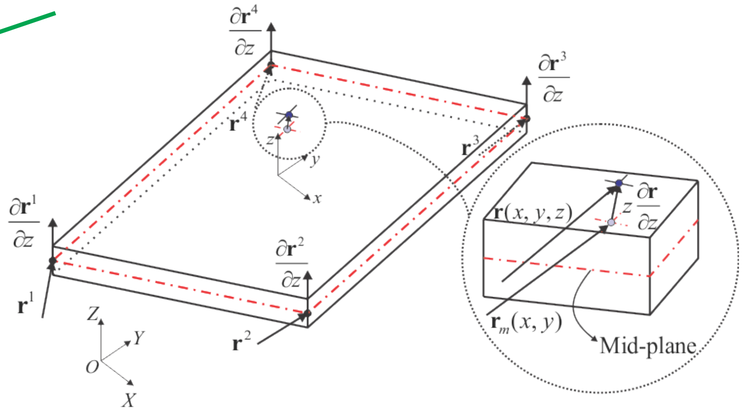
- Point forces
- Surface forces, e.g. evenly (or not) distributed pressure
- Volumetric forces

Generalized external forces: Point load

Point load:

- concentrated load
- acts on one finite element at any point
- does not require numerical integration

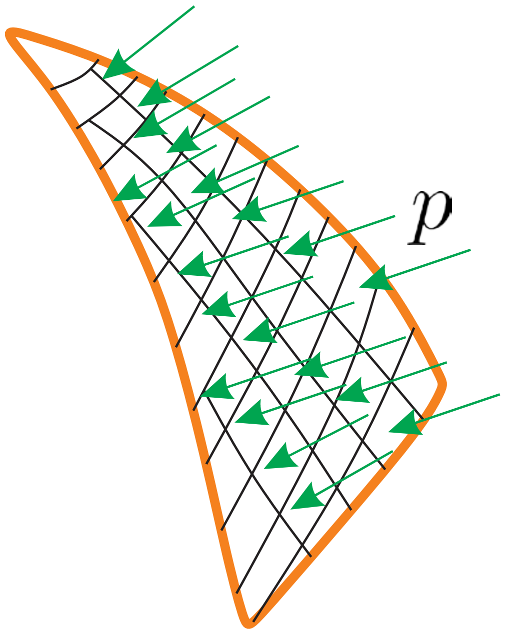




$$\delta W_{cl} = \mathbf{F}^T \delta \mathbf{r}^P = \underbrace{\mathbf{Q}_{cl}^T}_{\text{Generalized force}} \underbrace{\delta \mathbf{e}}_{\text{Variation of generalized forces}}$$

$$\mathbf{Q}_{cl} = \mathbf{S}^T \left(\xi_P, \eta_P\right) \mathbf{F}^P$$

Generalized external forces: Pressure



Pressure:

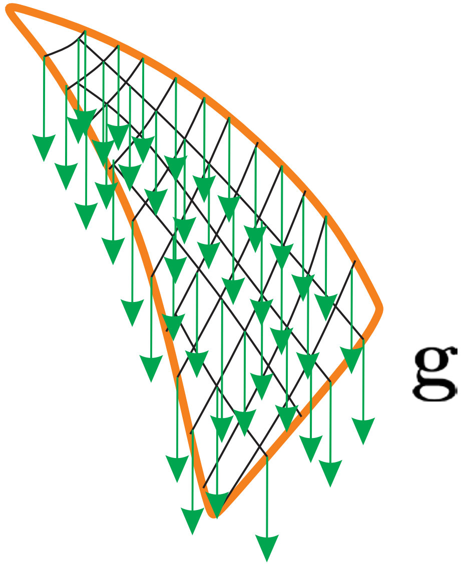
- distributed load
- acts normal to the surface
- use Principle of Virtual Work to obtain generalized counterpart

Normal definition: dependent on element's kinematics

$$\mathbf{Q}_{pres} = -\int_A \underbrace{\mathbf{S}^T(\xi,\eta)}_{\text{Shape function}} \overbrace{p}^{\text{constant pressure}} \underbrace{\boxed{\mathbf{n}}}_{\text{normal to the surface}} \overbrace{\det[\mathbf{J}]}^{\text{Det. of Jacobian of the transformation}} \mathrm{d}A = \overbrace{\sum_{j=0}^{n_j}\sum_{i=0}^{n_i} w_i w_j \mathbf{S}^T(\xi_i,\eta_j) p\mathbf{n}(\xi_i,\eta_j)\det[\mathbf{J}]}^{\text{Numerically solve the integral: Gauss Quadrature}}$$

Animation: ANCF Shell + Initial Configuration + Internal Pressure

Generalized external forces: Gravity load



Gravity load:

- volumetric, distributed load
- acts along a global direction
- do not depend on the finite element's coordinates

$$\mathbf{Q}_{pres} = -\int_V \underbrace{\mathbf{S}^T(\xi,\eta,\zeta)}_{\text{Shape function}} \overbrace{\rho}^{\text{density}} \underbrace{\mathbf{g}}_{\text{acceleration of gravity}} \overbrace{\det[\mathbf{J}]}^{\text{Det. of Jacobian of the transformation}} dV = \overbrace{\sum_{k=0}^{n_k}\sum_{j=0}^{n_j}\sum_{i=0}^{n_i} w_i w_j w_k \mathbf{S}^T(\xi_i,\eta_j,\zeta_k)\rho\mathbf{g}\det[\mathbf{J}]}^{\text{Numerically solve the integral: Gauss Quadrature}}$$

ANCF in Chrono

Chrono::FEA::ANCF ecosystem still growing

- 3-node shear deformable beam element implemented in the last month
- New higher-order shell element underway