General Theoretical Concepts Related to Multibody Dynamics
Before Getting Started

• Material draws on two main sources


  • Course notes, available at: http://sbel.wisc.edu/Courses/ME751/2016/
Looking Ahead

• Purpose of this segment:
  • Quick discussion of several theoretical concepts that come up time and again when using Chrono

• Concepts covered
  • Reference frames and changes of reference frames
  • Elements of the kinematics of a 3D body (position, velocity and acceleration of a body)
  • Kinematic constraints (joints)
  • Formulating the equations of motion
    • Newton-Euler equations of motion (via D’Alembert’s Principle)
Reference Frames in 3D Kinematics. Problem Setup

• Global Reference Frame (G-RF) attached to ground at point O

• Imagine point P is fixed (red-pen mark) on the rigid body

• Rigid body has a reference frame attached (fixed) to it
  • Assume its origin is at O (same as G-RF)
  • Called Local Reference Frame (L-RF) – shown in blue
  • Axes: \( f, g, h \)

• Question of interest:
  • What is the relationship between the coordinates of point P in G-RF and L-RF?
More Formal Way of Posing the Question

- Let $\vec{q} = \overrightarrow{OP}$ be a geometric vector (see figure)

- In the G-RF defined by $(\vec{i}, \vec{j}, \vec{k})$, the geometric vector $\vec{q}$ is represented as

  $$\vec{q} = q_x \vec{i} + q_y \vec{j} + q_z \vec{k}$$

- In the L-RF defined by $(\vec{f}, \vec{g}, \vec{h})$, the geometric vector $\vec{q}$ is represented as

  $$\vec{q} = \bar{q}_x \vec{f} + \bar{q}_y \vec{g} + \bar{q}_z \vec{h}$$

- QUESTION: how are $(q_x, q_y, q_z)$ and $(\bar{q}_x, \bar{q}_y, \bar{q}_z)$ related?
Relationship Between L-RF Vectors and G-RF Vectors

\[
\vec{f} = a_{11} \vec{i} + a_{21} \vec{j} + a_{31} \vec{k}
\]

\[
\vec{g} = a_{12} \vec{i} + a_{22} \vec{j} + a_{32} \vec{k}
\]

\[
\vec{h} = a_{13} \vec{i} + a_{23} \vec{j} + a_{33} \vec{k}
\]

\[
f = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \quad g = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \quad h = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}
\]

\[
a_{11} = \vec{i} \cdot \vec{f} = \cos \theta(\vec{i}, \vec{f})
\]

\[
a_{12} = \vec{i} \cdot \vec{g} = \cos \theta(\vec{i}, \vec{g})
\]

\[
a_{13} = \vec{i} \cdot \vec{h} = \cos \theta(\vec{i}, \vec{h})
\]

\[
a_{21} = \vec{j} \cdot \vec{f} = \cos \theta(\vec{j}, \vec{f})
\]

\[
a_{22} = \vec{j} \cdot \vec{g} = \cos \theta(\vec{j}, \vec{g})
\]

\[
a_{23} = \vec{j} \cdot \vec{h} = \cos \theta(\vec{j}, \vec{h})
\]

\[
a_{31} = \vec{k} \cdot \vec{f} = \cos \theta(\vec{k}, \vec{f})
\]

\[
a_{32} = \vec{k} \cdot \vec{g} = \cos \theta(\vec{k}, \vec{g})
\]

\[
a_{33} = \vec{k} \cdot \vec{h} = \cos \theta(\vec{k}, \vec{h})
\]

There is a good reason the values \(a_{ij}\) above are called “direction cosines”.

6
Punch Line, Change of Reference Frame (from "source" to "destination")

\[
\begin{bmatrix}
q_x \\
q_y \\
q_z
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
\bar{q}_x \\
\bar{q}_y \\
\bar{q}_z
\end{bmatrix}
\]

\[q_d = A_{ds} q_s\]

\[A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
= \begin{bmatrix}
f \\
g \\
h
\end{bmatrix}\]

\[f = \begin{bmatrix}
a_{11} \\
a_{21} \\
a_{31}
\end{bmatrix}
\quad g = \begin{bmatrix}
a_{12} \\
a_{22} \\
a_{32}
\end{bmatrix}
\quad h = \begin{bmatrix}
a_{13} \\
a_{23} \\
a_{33}
\end{bmatrix}\]
The Bottom Line: Moving from RF to RF

- Representing the same geometric vector in two different RFs leads to the concept of “rotation matrix”, or “transformation matrix” $A_{ds}$:

  - Getting the new coordinates, that is, representation of the same geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix $A_{ds}$:

    $$q = A_{ds} \tilde{q}$$

- NOTE 1: what is changed is the RF used to represent the vector
  - We are talking about the *same* geometric vector, represented in two RFs

- NOTE 2: rotation matrix $A_{ds}$ sometimes called “orientation matrix”
Rotation Matrix is Orthogonal

- Recall that \( \vec{f}, \vec{g}, \) and \( \vec{h} \) are mutually orthogonal
- Recall that \( \vec{f}, \vec{g}, \) and \( \vec{h} \) are unit vectors
- Therefore, the following holds:

\[
\begin{align*}
\vec{f}^T \vec{f} &= \vec{g}^T \vec{g} = \vec{h}^T \vec{h} = 1 \\
\vec{f}^T \vec{g} &= \vec{g}^T \vec{h} = \vec{h}^T \vec{f} = 0
\end{align*}
\]

- Consequently, the rotation matrix \( \mathbf{A} \) is orthogonal

\[
\mathbf{A}^T \mathbf{A} = \mathbf{A} \mathbf{A}^T = \mathbf{I}_{3 \times 3}
\]
Summarizing Key Points, Reference Frames

- Started with the representation $q_s$ of a geometric vector $\mathbf{q}$ in a “source” reference frame $s$

- The representation of the geometric vector $\mathbf{q}$ in a “destination” reference frame $d$ is given by

$$q_d = A_{ds}q_s$$

- Matrix $A_{ds}$ called transformation, or rotation matrix (taking vector from the source RF $s$ to the destination RF $d$)

- Because $A_{ds}$ is orthogonal, one has that

$$q_s = A_{ds}^T q_d \quad \text{therefore} \quad A_{sd} = A_{ds}^T$$

- Many times, the “destination” RF is the global reference frame (G-RF), which has ID “0”
  - In this case, we don’t show “0” anymore, simply call $A_s$ instead of $A_{0s}$
New Topic:
Angular Velocity. 3D Problem Setup

• Global Reference Frame (G-RF) attached to ground at point O

• Imagine point P is fixed (red-pen mark) on the rigid body

• Rigid body has a reference frame attached to it
  • Assume its origin is at O (same as G-RF)
  • Local Reference Frame (L-RF) – shown in blue
  • Axes: \( \mathbf{f}, \mathbf{g}, \mathbf{h} \)

• Question of interest:
  • How do we express rate of change of blue RF wrt global RF?
Angular Velocity, Getting There...

- Recall that \( \mathbf{A}_i \mathbf{A}_i^T = \mathbf{I}_{3 \times 3} \). Taking a time derivative yields
  \[
  \dot{\mathbf{A}}_i \mathbf{A}_i^T + \mathbf{A}_i \dot{\mathbf{A}}_i^T = 0_{3 \times 3} \quad \Rightarrow \quad \dot{\mathbf{A}}_i \mathbf{A}_i^T = -\mathbf{A}_i \dot{\mathbf{A}}_i^T
  \]

- Quick remarks
  - The matrix \( \dot{\mathbf{A}}_i \mathbf{A}_i^T \) is a \( 3 \times 3 \) matrix
  - The matrix \( \dot{\mathbf{A}}_i \mathbf{A}_i^T \) is skew-symmetric

- CONCLUSION: there must be a vector, \( \omega_i \), whose cross product matrix is equal to the \( 3 \times 3 \) skew symmetric matrix \( \dot{\mathbf{A}}_i \mathbf{A}_i^T \):
  \[
  \ddot{\omega}_i = \dot{\mathbf{A}}_i \mathbf{A}_i^T
  \]

- This vector \( \omega_i \) is called the angular velocity of the L-RF with respect to the G-RF.
Angular Velocity: Represented in G-RF or in L-RF

- Since $A_i$ is orthogonal, rate of change $\dot{A}_i$ of orientation matrix is simply
  \[ \dot{A}_i = \tilde{\omega}_i A_i \]

- Angular velocity vector can be represented in the local reference frame. Skipping details,
  \[ \tilde{\omega}_i = A_i^T \dot{A}_i \]

- Therefore, rate of change $\dot{A}_i$ of orientation matrix can also be represented as
  \[ \dot{A}_i = A_i \tilde{\omega}_i \]

- Notation convention: an over-bar placed on a vector (like $\tilde{\omega}_i$ above) indicates that quantity is a representation of a geometric vector in a local reference frame
New Topic: Using Euler Parameters to Define Rotation Matrix $A$

- **Starting point: Euler’s Theorem**
  
  “If the origins of two right-hand Cartesian reference frames coincide, then the RFs may be brought into coincidence by a single rotation of a certain angle $\chi$ about a carefully chosen unit axis $\mathbf{u}$”

- **Euler’s Theorem proved in the following references:**
  - Wittenburg – Dynamics of Systems of Rigid Bodies (1977)
Warming up…

- Green color - used for quantities that define the Euler rotation: the axis of rotation defined by the **unit** vector $\mathbf{u}$ and the angle $\chi$

- Red color - used to indicate the vectors that need to be summed up to get axis $\mathbf{h}$ of the L-RF

- Blue color - denotes the $\mathbf{f} - \mathbf{g} - \mathbf{h}$ axes of the L-RF

- Black dotted line - support entities (helpers, don’t play any role but only help with the derivation). The angle $\alpha$ measured between the axis of rotation $\mathbf{u}$ and the $\mathbf{k}$ unit vector.

- Other notation used: $||\mathbf{a}|| = a$ \quad $||\mathbf{b}|| = b$ \quad $||\mathbf{c}|| = c$
How Euler Parameters Come to Be

- Using as input $\chi$ and $u$, one can express the vectors $\vec{f}$, $\vec{g}$, and $\vec{h}$ in the global reference frame as

$$
\begin{align*}
\vec{f} &= i(2\cos^2\frac{\chi}{2} - 1) + 2u(u^T i)\sin^2\frac{\chi}{2} + 2\dot{u}i\sin\frac{\chi}{2}\cos\frac{\chi}{2}, \\
\vec{g} &= j(2\cos^2\frac{\chi}{2} - 1) + 2u(u^T j)\sin^2\frac{\chi}{2} + 2\dot{u}j\sin\frac{\chi}{2}\cos\frac{\chi}{2}, \\
\vec{h} &= k(2\cos^2\frac{\chi}{2} - 1) + 2u(u^T k)\sin^2\frac{\chi}{2} + 2\dot{u}k\sin\frac{\chi}{2}\cos\frac{\chi}{2}
\end{align*}
$$

- The expression of $\vec{f}$, $\vec{g}$, and $\vec{h}$ justifies the introduction of the following generalized coordinates (the “Euler Parameters”):

$$
p = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad \text{where} \quad e_0 = \cos\frac{\chi}{2} \quad \text{and} \quad e \equiv \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = u\sin\frac{\chi}{2}
$$

- Note: $u$ unit vector $\Rightarrow$ values of $e_0$, $e_1$, $e_2$, and $e_3$ must satisfy the normalization condition

$$
e_0^2 + e_1^2 + e_2^2 + e_3^2 = e_0^2 + e^T e = 1$$
Orientation Matrix, Based on Euler Parameters

- Based on definition of $e_0, e_1, e_2,$ and $e_3,$

\[
\begin{align*}
  f &= [(2e_0^2 - 1)I + 2(ee^T + e_0\tilde{e})]i \\
  g &= [(2e_0^2 - 1)I + 2(ee^T + e_0\tilde{e})]j \\
  h &= [(2e_0^2 - 1)I + 2(ee^T + e_0\tilde{e})]k
\end{align*}
\]

- Recall that $A = [f \ g \ h]$ 
- Therefore,

\[
A = [(2e_0^2 - 1)I + 2(ee^T + e_0\tilde{e})]
\]

- Equivalently,

\[
A = 2 \begin{bmatrix}
  e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\
  e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\
  e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2}
\end{bmatrix}
\]
So far, focus was only on the rotation of a rigid body

Body connected to ground through a spherical joint
- Body experienced an arbitrary rotation

Yet bodies are experiencing both translation and rotation
3D Kinematics of Rigid Body: Problem Backdrop

- Framework and Notation Conventions:
  - A L-RF is attached to the rigid body at some location denoted by $O'$
  - Relative to the G-RF, point $O'$ is located by vector $\vec{r}$
  - L-RF defined by vectors $\vec{f}$, $\vec{g}$, $\vec{h}$
  - An arbitrary point $P$ of the rigid body is considered. Its location relative to the L-RF is provided through the vector $\vec{s}^P$
3D Rigid Body Kinematics: Position of an Arbitrary Point \( P \)

- The Geometric View:
  \[
  \overrightarrow{OP} = \overrightarrow{O'O} + \overrightarrow{O'P}
  \]
  \[
  \vec{r}^P = \vec{r} + \vec{s}^P
  \]

- The Algebraic Representation:
  \[
  \vec{r}^P = \vec{r} + \vec{s}^P = \vec{r} + A\vec{s}^P
  \]

- Important observation:
  - The vector \( \vec{s}^P \) that provides the location of \( P \) in the L-RF is a constant vector
    * True because the body is assumed to be rigid
3D Rigid Body Kinematics: Velocity of Arbitrary Point P

• In the Geometric Vector world:

\[ \vec{v}^P = \frac{d\vec{r}^P}{dt} = \dot{\vec{r}} + \dot{s}^P = \dot{\vec{r}} + \vec{\omega} \times \vec{s}^P \]

• Using the Algebraic Vector representation (Chrono):

\[ \dot{\vec{r}}^P = \dot{\vec{r}} + \dot{s}^P = \dot{\vec{r}} + \dot{\vec{A}}\vec{s}^P = \dot{\vec{r}} + \vec{\omega}A\vec{s}^P = \dot{\vec{r}} + \vec{\omega}s^P \]

• In plain words: the velocity \( \dot{\vec{r}}^P \) of a point P is equal to the sum of the velocity \( \dot{\vec{r}} \) of the point where the L-RF is located and the velocity \( \vec{\omega}s^P \) due to the rotation with angular velocity \( \vec{\omega} \) of the rigid body.
3D Rigid Body Kinematics: Acceleration of Arbitrary Point P

- In the Geometric Vector world, by definition:

\[
\ddot{\mathbf{r}}^P \equiv \frac{d^2 \mathbf{r}^P}{dt^2} = \ddot{\mathbf{r}} + \dot{\omega} \times \dot{\mathbf{r}}^P + \ddot{\omega} \times \mathbf{s}^P
\]

- Using the Algebraic Vector representation (Chrono):

\[
\mathbf{a}^P \equiv \ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \dot{s}^P = \ddot{\mathbf{r}} + \tilde{\omega} \tilde{\omega} \mathbf{s}^P + \tilde{\omega} \mathbf{\dot{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega} \tilde{\omega} \mathbf{s}^P + \tilde{\omega} \mathbf{s}^P
\]
Putting Things in Perspective: What We’ve Covered so Far

• Discussed how to get the expression of a geometric vector in a “destination” reference frame knowing its expression in a “source” reference frame
  • Done via rotation matrix A

• Euler Parameters: a way of computing the A matrix when knowing the axis of rotation and angle of rotation

• Rate of change of the orientation matrix A → led to the concept of angular velocity

• Position, velocity and acceleration of a point P attached to a rigid body
Looking Ahead

• Kinematic constraints; i.e., joints

• Formulating the equations of motion
New Topic:
Kinematic Constraints

• Geometric Constraint (GCon): a real world geometric attribute of the motion of the mechanical system
  • Examples:
    • Particle moves around point (1,2,3) on a sphere of radius 2.0
    • A unit vector $\mathbf{u}_6$ on body 6 is perpendicular on a certain unit vector $\mathbf{u}_9$ on body 9
    • The $y$ coordinate of point Q on body 8 is 14.5

• Algebraic Constraint Equations (ACEs): in the virtual world, a collection of one or more algebraic constraints, involving the generalized coordinates of the mechanism and possibly time $t$, that capture the geometry of the motion as induced by a certain Geometric Constraint
  • Examples:
    • $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 - 4 = 0$
    • $\mathbf{u}_6^T \cdot \mathbf{u}_9 = 0$
    • $[0 \ 1 \ 0] \cdot r_Q - 14.5 = 0$

• Modeling: the process that starts with the idealization of the real world to yield a GCon and continues with the GCon abstracting into a set of ACEs
Basic Geometric Constraints (GCons)

- We have four basic GCons:
  - DP1: the dot product of two vectors on two bodies is specified
  - DP2: the dot product of a vector of on a body and a vector between two bodies is specified
  - D: the distance between two points on two different bodies is specified
  - CD: the difference between the coordinates of two bodies is specified

- Note:
  - DP1 stands for Dot Product 1
  - DP2 stands for Dot Product 2
  - D stands for distance
  - CD stands for coordinate difference
Basic GCon: DP1

- Geometrically:
  \[ \mathbf{a}_i \cdot \mathbf{a}_j - f(t) = 0 \]

- Algebraically (matrix-vector notation):
  \[ \Phi_{DP1}(i, \mathbf{a}_i, j, \mathbf{a}_j, f(t)) = \mathbf{a}_i^T A_i^T A_j \mathbf{a}_j - f(t) = 0 \]
Basic GCon: DP2

- Geometrically:
  \[ \vec{a}_i \cdot \vec{d}_{ij} - f(t) = 0 \]

- Algebraically (matrix-vector notation):
  \[
  \Phi^{DP2}(i, \vec{a}_i, \vec{s}_i^P, j, \vec{s}_j^Q, f(t)) = \vec{a}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = 0
  \]
  \[
  = \vec{a}_i^T \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \vec{s}_j^Q - \mathbf{r}_i - \mathbf{A}_i \vec{s}_i^P) - f(t) = 0
  \]
Basic GCon: D

- Geometrically:
  \[ \vec{d}_{ij} \cdot \vec{d}_{ij} - f^2(t) = 0 \]

- Algebraically (matrix-vector notation):
  \[ \Phi^D(i, \vec{s}_i^P, j, \vec{s}_j^Q, f(t)) = \begin{pmatrix} d_{ij} \\ \vdots \end{pmatrix}^T \begin{pmatrix} d_{ij} \\ \vdots \end{pmatrix} - f^2(t) \]
  \[ = (\vec{r}_j + A_j \vec{s}_j^Q - \vec{r}_i - A_i \vec{s}_i^P)^T (\vec{r}_j + A_j \vec{s}_j^Q - \vec{r}_i - A_i \vec{s}_i^P) - f^2(t) = 0 \]
Basic GCon: CD

- Geometrically (\( \mathbf{c} \) is a constant vector):

\[
\mathbf{c} \cdot (\mathbf{a}_j - \mathbf{a}_i) - f(t) = 0
\]

- Algebraically (matrix-vector notation):

\[
\Phi^{CD}(\mathbf{c}, i, \mathbf{s}_i^P, j, \mathbf{s}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = \mathbf{c}^T (\mathbf{r}_j + \mathbf{A}_j \mathbf{s}_j^Q - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}_i^P) - f(t) = 0
\]
Intermediate GCons

• Two Intermediate GCons:
  • \( \perp_1 \): a vector is perpendicular on a plane belonging to a different body
  • \( \perp_2 \): a vector between two bodies is perpendicular to a plane belonging to the different body
Intermediate GCon: ⊥ 1 (Perpendicular Type 1)

- Geometrically, the motion is such that a vector \( \mathbf{c}_j \) on body \( j \) is perpendicular to a plane of body \( i \) that is defined by \( \mathbf{a}_i \) and \( \mathbf{b}_i \)

- Algebraically (matrix-vector notation):

\[
\Phi^{-1}(i, \bar{a}_i, \bar{b}_i, j, \bar{c}_j) = 
\begin{bmatrix}
\Phi^{DP1}(i, \bar{a}_i, j, \bar{c}_j, 0) \\
\Phi^{DP1}(i, \bar{b}_i, j, \bar{c}_j, 0)
\end{bmatrix} = 
\begin{bmatrix}
\bar{a}_i^T A_i^T A_j \bar{c}_j \\
\bar{b}_i^T A_i^T A_j \bar{c}_j
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
Intermediate GCon: $\perp 2$ (Perpendicular Type 2)

- Geometrically, a vector $\overrightarrow{P_iQ_j}$ from body $i$ to body $j$ remains perpendicular to a plane defined by two vectors $\tilde{a}_i$ and $\tilde{b}_i$

- Algebraically (matrix-vector notation):

$$\Phi_{\perp 2}^{DP2}(i, \tilde{a}_i, \tilde{b}_i, \bar{s}_i^P, j, \bar{s}_j^Q) = \begin{bmatrix} \Phi_{DP2}^{DP2}(i, \tilde{a}_i, \bar{s}_i^P, j, \bar{s}_j^Q, 0) \\
\Phi_{DP2}^{DP2}(i, \tilde{b}_i, \bar{s}_i^P, j, \bar{s}_j^Q, 0) \end{bmatrix} = \begin{bmatrix} \tilde{a}_i^T A_i^T d_{ij} \\
\tilde{b}_i^T A_i^T d_{ij} \end{bmatrix} = 0$$
High Level GCons

- High Level GCons also called joints:
  - Spherical Joint (SJ)
  - Universal Joint (UJ)
  - Cylindrical Joint (CJ)
  - Revolute Joint (RJ)
  - Translational Joint (TJ)
  - Other composite joints (spherical-spherical, translational-revolute, etc.)
High Level GCon: SJ [Spherical Joint]

\[
\Phi^{SJ} = \begin{bmatrix}
\Phi^{CD}(i, i, \bar{s}_i^P, j, \bar{s}_j^Q, 0) \\
\Phi^{CD}(j, i, \bar{s}_i^P, j, \bar{s}_j^Q, 0) \\
\Phi^{CD}(k, i, \bar{s}_i^P, j, \bar{s}_j^Q, 0)
\end{bmatrix}
\]
High Level GCon: CJ [Cylindrical Joint]
High Level GCon: TJ [Translational Joint]

\[
\Phi^{TJ} = \begin{bmatrix}
\Phi^{CJ}(i, s_i^P, \bar{a}_i, \bar{b}_i, j, s_j^Q, c_j) \\
\Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, \text{const.})
\end{bmatrix}
\]
High Level GCon: RJ [Revolute Joint]

\[
\Phi^{RJ} = \begin{bmatrix}
\Phi^{SJ}(i, s^P_i, j, s^Q_j) \\
\Phi^{\perp 1}(i, \tilde{a}_i, \tilde{b}_j, j, \tilde{c}_j)
\end{bmatrix}
\]
High Level GCon: UJ [Universal Joint]

$$\Phi_{UJ} = \begin{bmatrix} \Phi^{SJ}(i, \tilde{s}^P_i, j, \tilde{s}^Q_j) \\ \Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, 0) \end{bmatrix}$$

Figure 9.4.15  Singular behavior of universal joint.
### Connection Between Basic and Intermediate/High Level GCons

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<th>$DP_1$</th>
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- Note that there are other GCons that are used, but they see less mileage.
Constraints Supported in Chrono
New Topic:
Formulating the Equations of Motion

• Road map, full derivation of constrained equations of motion
  
  • Step 1: Introduce the types of force acting on one body present in a mechanical system
    • Distributed
    • Concentrated
  
  • Step 2: Express the virtual work produced by each of these forces acting on one body
  
  • Step 3: Evaluate the virtual work for the entire mechanical system
  
  • Step 4: Apply principle of virtual work (via D’Alembert’s principle) to obtain the EOM
Generic Forces/Torques Acting on a Mechanical System

• Distributed forces
  • Inertia forces
  • Volume/Mass distributed force (like gravity, electromagnetic, etc.)
  • Internal forces

• Concentrated forces/torques
  • Reaction forces/torques (induces by the presence of kinematic constraints)
  • Externally applied forces and torques (me pushing a cart)
Virtual Work for One Body, Side Trip

- Quick example below only shows virtual work produced by the **inertial force**
  - Same recipe applied for all other forces, distributed or concentrated

- Starting point: consider point $P$ of body $i$ associated with infinitesimal mass element $dm_i(P)$

- Expression of the force:
  \[-\ddot{r}_i^P \, dm_i(P)\]

- Virtual work produced:
  \[
  [\delta \mathbf{r}_i^P]^T \cdot [\ddot{r}_i^P \, dm_i(P)]
  \]

- Body virtual work obtained by summing over all points $P$ of body $i$:
  \[
  \delta W = \int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{r}_i^P \, dm_i(P)
  \]

- Upon expressing virtual displacement of $P$ and its acceleration $\ddot{r}_i^P$:
  \[
  \delta W = \int_{m_i} [\delta \mathbf{r}_i^T + \delta \pi_i^T \mathbf{s}_i^P \mathbf{A}_i^T] \cdot [\ddot{r}_i + \mathbf{A}_i \ddot{\omega}_i \mathbf{s}_i^P + \mathbf{A}_i \dddot{\omega}_i \mathbf{\dddot{s}}_i^P] \, dm_i(P) = \delta \mathbf{r}_i^T \, m_i \ddot{r}_i + \delta \pi_i^T [\dddot{\omega}_i \mathbf{J}_i \ddot{\omega}_i + \mathbf{\dddot{J}}_i \dot{\omega}_i]}
  \]
Final Form, Expression of Virtual Work

- When all said and done, the expression of the virtual work assumes the form:

\[
\delta W = \sum_{i=1}^{nb} \left[ -\delta \mathbf{r}_i^T \mathbf{m}_i \ddot{\mathbf{r}}_i - \delta \mathbf{\bar{n}}_i^T \ddot{\mathbf{\bar{\imath}}} i \mathbf{\bar{J}}_i \dot{\mathbf{\bar{\omega}}}_i - \delta \mathbf{\bar{n}}_i^T \mathbf{\bar{J}}_i \dot{\mathbf{\bar{\omega}}}_i + \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \mathbf{\bar{n}}_i^T \cdot \mathbf{\bar{n}}_i^m \\
+ \delta \mathbf{r}_i^T \mathbf{F}_i^a + \delta \mathbf{\bar{n}}_i^T \mathbf{\bar{n}}_i^a + \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \mathbf{\bar{n}}_i^T \mathbf{\bar{n}}_i^r \right] = 0
\]

- Alternatively,

\[
\delta W = \sum_{i=1}^{nb} \left[ \delta \mathbf{r}_i^T \left( -m_i \ddot{\mathbf{r}}_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \right) + \delta \mathbf{\bar{n}}_i^T \left( -\ddot{\mathbf{\bar{\imath}}} i \mathbf{\bar{J}}_i \dot{\mathbf{\bar{\omega}}}_i - \mathbf{\bar{J}}_i \dot{\mathbf{\bar{\omega}}}_i + \mathbf{\bar{n}}_i^m + \mathbf{\bar{n}}_i^a + \mathbf{\bar{n}}_i^r \right) \right] = 0
\]
Moving from One Body to a Mechanical System

- Total virtual work, for the entire system, assumes the form:

\[
\delta W = \sum_{i=1}^{nb} \left[ -\delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i m_i - \delta \pi_i^T \ddot{\mathbf{\pi}}_i \mathbf{I}_i \ddot{\mathbf{\omega}}_i - \delta \mathbf{\pi}_i^T \mathbf{J}_i \dot{\mathbf{\omega}}_i + \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \pi_i^T \cdot \mathbf{n}_i^m \\
+ \delta \mathbf{r}_i^T \mathbf{F}_i^a + \delta \pi_i^T \mathbf{n}_i^a + \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \mathbf{\pi}_i^T \mathbf{n}_i^r \right] = 0
\]

- Alternatively,

\[
\delta W = \sum_{i=1}^{nb} \left[ \delta \mathbf{r}_i^T \left( -\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \right) + \delta \pi_i^T \left( -\ddot{\mathbf{\pi}}_i \mathbf{I}_i \ddot{\mathbf{\omega}}_i - \mathbf{J}_i \dot{\mathbf{\omega}}_i + \mathbf{n}_i^m + \mathbf{n}_i^a + \mathbf{n}_i^r \right) \right] = 0
\]

- Recall that for each body \( i \), virtual translations \( \delta \mathbf{r}_i \) and virtual rotations \( \delta \pi_i \) are arbitrary
Equations of Motion (EOM) for A System of Rigid Bodies

- Since equation on previous slide should hold for any set of virtual displacements \((\delta r_1, \delta \pi_1), (\delta r_2, \delta \pi_2), \ldots, (\delta r_{nb}, \delta \pi_{nb})\), then we necessarily have that for \(i = 1, \ldots, nb\):
  \[
  -m_i \ddot{r}_i + F^m_i + F^a_i + F^r_i = 0_3 \\
  -\ddot{\omega}_i \dddot{\omega}_i - \dddot{J}_i \dot{\omega}_i + \dddot{n}_i^m + \dddot{n}_i^a + \dddot{n}_i^r = 0_3
  \]

- Equivalently, for \(i = 1, \ldots, nb\)
  \[
  m_i \ddot{r}_i = F^m_i + F^a_i + F^r_i \\
  \dddot{J}_i \dot{\omega}_i = \dddot{n}_i^m + \dddot{n}_i^a + \dddot{n}_i^r - \dddot{\omega}_i \dddot{J}_i \dot{\omega}_i
  \]

- The set of equations above represent the EOM for the system of \(nb\) rigid bodies.
The Joints (Kinematic Constraints) Lead to Reaction Forces

- The collection of all \( nc \) kinematic and driving constraints – stack them together:

\[
\Phi(q, t) = \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q, t) \end{bmatrix} = 0_{nc}
\]

- Recall that any one of the constraints in \( \Phi \) is one of the four basic GCons introduced earlier.

- The variation of \( \Phi \): stack together the variation of each of the GCons that enters in \( \Phi \).

- A virtual displacement of the bodies in the system will lead to a virtual variation \( \delta \Phi \) that depends on the position and orientation of the bodies:

\[
\delta \Phi = \Phi_r \delta r + \Pi(\Phi) \delta \pi = 0_{nc}
\]

- In matrix form, we can express the above relations as

\[
\delta \Phi(r, p) = [ \Phi_r \quad \Pi(\Phi) ] \cdot \begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix} = \mathcal{R}(\Phi) \cdot \begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix} = 0_{nc}
\]

- \( \Phi_r \) and \( \Pi(\Phi) \): the key ingredients needed to express the reaction forces induced by the constraints \( \Phi(q, t) = 0_{nc} \).
Switching to Matrix-Vector Notation

- Notation used to simplify expression of EOM:
  - $I_3$ is the identity matrix of dimension 3
  - $F^a_i$ and $F^m_i$ – applied and mass-distributed force, body $i$
  - $\vec{n}^a_i$ and $\vec{n}^m_i$ – applied and mass-distributed torque, body $i$
  - $m_i$ and $\vec{J}_i$ – mass and mass moment of inertia, body $i$

- Matrix-vector notation:

$$M = \begin{bmatrix}
m_1 I_3 & 0_{3\times 3} & \ldots & 0_{3\times 3} \\
0_{3\times 3} & m_2 I_3 & \ldots & 0_{3\times 3} \\
\vdots & \vdots & \ddots & \vdots \\
0_{3\times 3} & 0_{3\times 3} & \ldots & m_{nb} I_3
\end{bmatrix} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
EOM: the Newton-Euler Form

- According to Lagrange Multiplier theorem, there exists a vector of Lagrange Multipliers, \( \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{nc} \end{bmatrix} \), so that

\[
\begin{bmatrix}
\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} \\
\mathbf{J}\ddot{\omega} - \tau
\end{bmatrix}
+ \begin{bmatrix}
\Phi_r^T \\
\Pi^T(\Phi)
\end{bmatrix} \lambda = \mathbf{0}_{6nb}
\]

- Expression above: \textbf{Newton-Euler form of the EOM}. Equivalently expressed as:

\[
\begin{cases}
\mathbf{M}\ddot{\mathbf{r}} + \Phi_r^T \lambda = \mathbf{F} \\
\mathbf{J}\ddot{\omega} + \Pi^T(\Phi)\lambda = \tau
\end{cases}
\]